# Week 2

## Combinatorics

1. (2018B6) Let $S$ be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of $S$ is at most

$$2^{3860}\left(\frac{2018}{2048}\right)^{2018}.$$

2. Find the number of permutations $\sigma$ on $\{1, \ldots, n\}$ such that

$$s(\sigma) := \sum_{k=1}^{n} |\sigma(k) - k| = \frac{n^2 - 1}{2}.$$

## Mock Putnam problems

A1 Let $f : (0, 1) \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = \int_0^1 x^2 f(x)dx = 0.$$

Prove that there exists distinct $a, b, c \in (0, 1)$ such that $f(a) = f(b) = f(c) = 0$.

A2 Let $p$ be a prime. Let $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 2a_n - pa_{n-1}$ for $n \geq 1$. For which value of $p$ does $-1$ appear as some $a_n$?

A4 Imagine the following game of life. Start with 1 cell. Every 1 second, each cell splits into three cells or two cells or stays one cell or dies, each with probability 1/4. What's the probability that eventually everything dies?

A6 (2020 B6) Define a sequence by $a_n = \lfloor n(\sqrt{10} - 3) \rfloor$. Prove that $\sum_{k=1}^{n}(-1)^{a_k} \geq 0$ for all $n \in \mathbb{N}$.

## Combinatorics

1. (2018B6) Let $S$ be the set of sequences of length 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of $S$ is at most

$$2^{3860}\left(\frac{2018}{2048}\right)^{2018}.$$

The cardinality of $S$ is $a_{3840}$ where

$$P(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6 + x^{10})^{2018} = \sum a_n x^n.$$

Since each $a_n \geq 0$, we have for any $x > 0$ and any $n > 0$,

$$a_n x^n \leq P(x).$$

Since we are asked to prove a bound of the form $a_n \leq 2^n()^{2018}$, it suggests considering $x = 1/2$:

$$a_n \frac{1}{2^n} \leq P(\frac{1}{2}) = \left(\frac{2018}{2048}\right)^{2018}.$$

2. Find the number of permutations $\sigma$ on $\{1, \ldots, n\}$ such that

$$s(\sigma) := \sum_{k=1}^{n} |\sigma(k) - k| = \frac{n^2 - 1}{2}.$$

Note that if $\sigma$ works, then so does $j \mapsto n + 1 - \sigma(j)$. Let $T_k$ be the set of such $\sigma$ with $\sigma((n + 1)/2) = k$. Then $\#T_k = \#T_{n+1-k}$. Suppose $\sigma((n + 1)/2) = k \leq (n + 1)/2$. Then the largest $s(\sigma)$ can take is when

$$\sigma(1), \ldots, \sigma(\frac{n-1}{2}) \in \{\frac{n+3}{2}, \ldots, n\} \qquad \text{and} \qquad \sigma(\frac{n+3}{2}), \ldots, \sigma(n) \in \{1, \ldots, \frac{n+1}{2}\} \setminus \{k\}.$$

For such $\sigma$, we have

$$s(\sigma) = 2\left(\frac{n+3}{2} + \cdots + n\right) - 2\left(1 + \cdots + \frac{n-1}{2}\right) = \frac{n^2-1}{2}.$$

Hence, we see that the above condition is necessary and sufficient for $\sigma \in T_k$. We have

$$\#T_k = \left[\left(\frac{n-1}{2}\right)!\right]^2.$$

Multiplying this by $n$ possible values of $k$ gives the answer.

## Mock Putnam problems

A1 Let $f : (0,1) \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = \int_0^1 x^2 f(x)dx = 0.$$

Prove that there exists distinct $a, b, c \in (0,1)$ such that $f(a) = f(b) = f(c) = 0$.

Suppose $f$ is not identically 0. If $f$ has no sign changes, then this contradicts $\int_0^1 f(x)dx = 0$. If $f$ has a unique sign change at $x = a$, then this contradicts $\int_0^1 (x-a)f(x)dx = 0$. If $f$ has sign changes only at $x = a$ and $x = b$ with $a < b$, then this contradicts $\int_0^1 (x-a)(x-b)f(x)dx = 0$. Hence $f$ has at least 3 sign changes.

A2 Let $p$ be a prime. Let $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 2a_n - pa_{n-1}$ for $n \geq 1$. For which value of $p$ does $-1$ appear as some $a_n$?

Working mod $p$, we have $a_n \equiv 2^{n-1} \pmod{p}$. Working mod $p-1$, we have $a_n \equiv n \pmod{p-1}$. So $n \equiv -1 \pmod{p-1}$ which implies that $2^n \equiv 2^{-1} \pmod{p}$. Hence $-1 \equiv 2^{-2} \pmod{p}$ implying that $p \mid 5$.

A4 Imagine the following game of life. Start with 1 cell. Every 1 second, each cell splits into three cells or two cells or stays one cell or dies, each with probability 1/4. What's the probability that eventually everything dies?

Let $X_n$ be the number of cells alive at stage $n$. Then

$$\text{Prob}(X_{n+1} = 0) = \sum_{m=0}^{\infty} \frac{1}{4^m} \text{Prob}(X_n = m) = E\left(\left(\frac{1}{4}\right)^{X_n}\right).$$

Now let $\alpha_n \in (0,1]$ be any real number, we relate $E(\alpha_n^{X_n})$ to $E(\alpha_{n-1}^{X_{n-1}})$. Note that

$$X_n | X_{n-1} = \sum_{i=1}^{X_{n-1}} Y_i \qquad \text{where} \qquad Y_i = \text{Unif}(0,1,2,3).$$

So we have

$$E(\alpha_n^{X_n} | X_{n-1}) = \prod_{i=1}^{X_{n-1}} E(\alpha_n^{Y_i}) = \left(\frac{1}{4}(\alpha_n^3 + \alpha_n^2 + \alpha_n + 1)\right)^{X_{n-1}}.$$

Therefore, we have

$$E(\alpha_n^{X_n}) = E(\alpha_{n-1}^{X_{n-1}}) \qquad \text{where} \qquad \alpha_{n-1} = \frac{1}{4}(\alpha_n^3 + \alpha_n^2 + \alpha_n + 1).$$

Since $X_1 = 1$, we see that $E(\alpha_n^{X_n}) = \alpha_1$. To find $\alpha_1$ as $n \to \infty$, we define the sequence

$$b_1 = \frac{1}{4}, \qquad b_{n+1} = \frac{1}{4}(b_n^3 + b_n^2 + b_n + 1).$$

It should now be standard exercise to show that $b_n$ is monotone and converges to $\sqrt{2} - 1$.

A6 (2020 B6) Define a sequence by $a_n = \lfloor n(\sqrt{10} - 3) \rfloor$. Prove that $\sum_{k=1}^{n} (-1)^{a_k} \geq 0$ for all $n \in \mathbb{N}$.

For any $j \geq 0$, let $b_j$ be the number of $n \in \mathbb{N}$ such that $a_n = j$. Let $\gamma = \sqrt{10} + 3$. Then

$$b_j = \#\{n\colon j\gamma < n < (j+1)\gamma\} = \lfloor (j+1)\gamma \rfloor - \lfloor j\gamma \rfloor.$$

Now $\gamma = 6 + \gamma^{-1}$, so we have

$$b_j = 6 + \lfloor (j+1)\gamma^{-1} \rfloor - \lfloor j\gamma^{-1} \rfloor.$$

This is 7 only when there is an integer $k \in (j\gamma^{-1}, (j+1)\gamma^{-1})$, or equivalently, $j < k\gamma < j+1$. In other words,

$$b_j = \begin{cases} 7 & \text{if } j = \lfloor k\gamma \rfloor \text{ for some } k \geq 1, \\ 6 & \text{otherwise.} \end{cases}$$

We now prove the desired statement by induction on $n$. It suffices to consider the case $a_n$ odd and $a_{n+1} = a_n + 1$. Let $m = a_n \leq n/6$. Then we have

$$\sum_{i=1}^{n} (-1)^{a_i} \geq \sum_{j=0}^{m} b_j (-1)^j = \sum_{j=0}^{m} (b_j - 6)(-1)^j = \sum_{\lfloor k\gamma \rfloor \leq m} (-1)^{\lfloor k\gamma \rfloor} = \sum_{k < (m+1)/\gamma} (-1)^{a_k}.$$

Now $(m+1)/\gamma < (n/6+1)/6 < n$. So we are done by induction.