Week 8: Mock Putnam 8

1: Let n be a positive integer and for any $k = 1, \ldots, n$, let $P_k = x_1^k + \cdots + x_n^k$ and let σ_k be the k-th elementary symmetric polynomial in $x_1, \ldots, x_n$. Prove Newton's identity: for all $k = 1, \ldots, n$,

$$P_k - \sigma_1 P_{k-1} + \sigma_2 P_{k-2} - \cdots + (-1)^{k-1}\sigma_{k-1}P_1 + (-1)^k k\sigma_k = 0.$$

2: Let S be the set of continuously differentiable functions $f(x) : [0,1] \to \mathbb{R}$ such that $f(0) = 0$ and $f(1) = 1$. Find

$$\min_{f \in S} \int_0^1 (1 + x^2) f'(x)^2 \, dx.$$

3: For any positive integer n, let $\Omega(n)$ be the number of prime factors of n with multiplicity. That is, $\Omega(n) = n_1 + \cdots + n_k$ if $n = p_1^{n_1} \cdots p_k^{n_k}$ is the prime factorization of n. It is known that

$$\sum_{k=1}^{n} (\Omega(k) - \log\log n)^2 = O(n \log\log n).$$

Prove that

$$\lim_{n\to\infty} \frac{\#\{ab : a, b = 1, \ldots, n\}}{n^2} = 0.$$

4: Let n be a positive integer. Find the minimum of $\dfrac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$ over distinct positive integers $x_1, \ldots, x_n$.

5: Find all monic polynomials $f(x)$ with integer coefficients such that for any positive integer n, there exists a positive integer a_n such that $f(a_n) = 2^n$.

6: Let $f(x)$ be a polynomial with rational coefficients of degree at least 2. Let $f^{[k]}(x) = f(f^{[k-1]}(x))$ denote the k-th iterate of f. Prove that the set

$$S = \{a \in \mathbb{Q} : \forall k \in \mathbb{N}, a = f^{[k]}(a_k) \text{ for some } a_k \in \mathbb{Q}\}$$

is finite.

Week 8: Sketch of proofs

1: For convenience, denote $\sigma_0 = 1$ and $P_0 = k$. Let $f(T) = (1 - x_1 T) \cdots (1 - x_n T)$, and consider the power series $f'(T)/f(T)$. Expanding f yields

$$f(T) = \sum_{i=0}^{n} (-1)^i \sigma_i T^i,$$
$$f'(T) = \sum_{i=1}^{n} (-1)^i i \sigma_i T^{i-1}.$$

On the other hand, using product rule yields

$$\frac{f'(T)}{f(T)} = \sum_{i=1}^{n} \frac{-x_i}{1 - x_i T} = -\sum_{i=1}^{n} \sum_{j=0}^{\infty} x_i^{j+1} T^j = -\sum_{j=0}^{\infty} P_{j+1} T^j.$$

Thus we get

$$-\left(\sum_{i=0}^{n} (-1)^i \sigma_i T^i\right)\left(\sum_{j=0}^{\infty} P_{j+1} T^j\right) = \sum_{i=1}^{n} (-1)^i i \sigma_i T^{i-1}.$$

Now we take the T^{k-1}-coefficient:

$$-\sum_{i+j=k-1} (-1)^i \sigma_i P_{j+1} = (-1)^k k \sigma_k \iff \sum_{i+j=k-1} (-1)^i \sigma_i P_{j+1} + (-1)^k k \sigma_k = 0,$$

which rearranges to the desired equality

$$P_k - \sigma_1 P_{k-1} + \sigma_2 P_{k-2} - \cdots + (-1)^{k-1} \sigma_{k-1} P_1 + (-1)^k k \sigma_k = 0.$$

2: Answer. $4/\pi$.

We first show that for any $f \in S$,

$$\int_0^1 (1 + x^2) f'(x)^2 \, dx \geq \frac{4}{\pi}.$$

Indeed, by Cauchy-Schwarz inequality,

$$\left(\int_0^1 (1 + x^2) f'(x)^2 \, dx\right)\left(\int_0^1 \frac{1}{1 + x^2} \, dx\right) \geq \left(\int_0^1 |f'(x)| \, dx\right)^2 \geq \left|\int_0^1 f'(x) \, dx\right|^2 = |f(1) - f(0)|^2 = 1.$$

Thus

$$\int_0^1 (1 + x^2) f'(x)^2 \, dx \geq \left(\int_0^1 \frac{1}{1 + x^2} \, dx\right)^{-1} = (\arctan 1 - \arctan 0)^{-1} = \frac{4}{\pi}.$$

To obtain equality, we need the equality $(1+x^2)f'(x)^2 = c/(1+x^2)$ for some constant $c > 0$. That is, $(1+x^2)f'(x)$ is constant, or $f'(x) = c/(1+x^2)$ for some constant $c > 0$. Thus, we can take

$$f(x) = f(0) + C\int_0^x \frac{1}{1+x^2}\,dx = C\arctan x,$$

where C is a constant such that $f(1) = 1$. Here $C = 4/\pi$ works, so the value $4/\pi$ is attained when $f(x) = \dfrac{4}{\pi}\arctan x$.

3: From the given equation, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{\Omega(k)}{\log\log n} - 1\right)^2 = O\left(\frac{1}{\log\log n}\right) \xrightarrow{n\to\infty} 0,$$

which means that for any $\epsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{n}\#\left\{k \le n : \left|\frac{\Omega(k)}{\log\log n} - 1\right| \ge \epsilon\right\} = 0.$$

Due to the triangle inequality, for any $x, y \in \mathbb{R}$ such that $|x+y| \ge 2\epsilon$, we have either $|x| \ge \epsilon$ or $|y| \ge \epsilon$. Thus we get

$$\lim_{n\to\infty}\frac{1}{n^2}\#\left\{a, b \le n : \left|\frac{\Omega(a)+\Omega(b)}{\log\log n} - 2\right| \ge \epsilon\right\} = 0.$$

Now, for each $n > 1$, let $S_n = \{ab : 1 \le a, b \le n\}$. Note that Ω is additive, i.e. $\Omega(ab) = \Omega(a) + \Omega(b)$ for any $a, b \in \mathbb{N}$. Thus, the above yields

$$\lim_{n\to\infty}\frac{1}{n^2}\#\left\{k \in S_n : \left|\frac{\Omega(k)}{\log\log n} - 2\right| \ge \epsilon\right\} = 0.$$

On the other hand, for n large enough, if $\left|\dfrac{\Omega(k)}{\log\log n} - 2\right| < \epsilon$, then

$$\frac{\Omega(k)}{\log\log n^2} - 1 \ge \frac{(2-\epsilon)\log\log n}{\log\log n + \log 2} - 1 \ge \frac{1}{2}.$$

The second limit equality shows that the number of k satisfying this inequality is $o(n^2)$, so we get

$$\lim_{n\to\infty}\frac{1}{n^2}\#\left\{k \in S_n : \left|\frac{\Omega(k)}{\log\log n} - 2\right| < \epsilon\right\} = 0.$$

Combining the fourth equality with this equality yields $\lim\limits_{n\to\infty} |S_n|/n^2 = 0$.

Note. A more careful analysis actually shows that

$$\frac{\#\{ab : a, b \le n\}}{n^2} = O\left(\frac{1}{\log\log n}\right).$$

Indeed, from the given big-O estimate, we have two inequalities:

$$\begin{aligned}
\frac{1}{n^2}\sum_{k\in S_n}\left(\frac{\Omega(k)}{\log\log n^2}-1\right)^2 &\le \frac{1}{n^2}\sum_{k=1}^{n^2}\left(\frac{\Omega(k)}{\log\log n^2}-1\right)^2\\
&= O\left(\frac{1}{\log\log n^2}\right)\\
&= O\left(\frac{1}{\log\log n}\right)\\
\frac{1}{n^2}\sum_{k\in S_n}\left(\frac{\Omega(k)}{\log\log n}-2\right)^2 &\le \frac{1}{n^2}\sum_{a=1}^{n}\sum_{b=1}^{n}\left(\frac{\Omega(ab)}{\log\log n}-2\right)^2\\
&\le \frac{2}{n^2}\sum_{a=1}^{n}\sum_{b=1}^{n}\left(\left(\frac{\Omega(a)}{\log\log n}-1\right)^2+\left(\frac{\Omega(b)}{\log\log n}-1\right)^2\right)\\
&= \frac{4}{n}\sum_{k=1}^{n}\left(\frac{\Omega(k)}{\log\log n}-1\right)^2\\
&= O\left(\frac{1}{\log\log n}\right)
\end{aligned}$$

By the same method as in the solution above, for n large enough and for any k,

$$\max\left\{\left|\frac{\Omega(k)}{\log\log n}-2\right|,\left|\frac{\Omega(k)}{\log\log n^2}-1\right|\right\}\ge\frac{1}{4}.$$

As a result, summing the two inequalities give

$$\frac{|S_n|}{n^2}=O\left(\frac{1}{\log\log n}\right).$$

4: Answer. $\dfrac{n(n+1)}{2}$.

We claim a stronger statement: for any distinct positive integers $x_1,\ldots,x_n$,

$$x_1^3+\cdots+x_n^3\ge(x_1+\cdots+x_n)^2.$$

Equality is attained, for example, when $x_i=i$ for each $i\le n$. The fact that $x_1,\ldots,x_n$ are pairwise distinct yields $x_1+x_2+\ldots+x_n\ge 1+2+\cdots+n=\binom{n+1}{2}$, and the above claim would then yield the desired answer.

We now prove the claim by induction on n, with the base case $n=1$ obvious. We may also WLOG assume that $x_1<x_2<\ldots<x_n$. For the induction step, consider arbitrary positive integers $x_1<x_2<\ldots<x_{n+1}$. Then by induction hypothesis,

$$x_1^3+\cdots+x_{n+1}^3\ge(x_1+x_2+\ldots+x_n)^2+x_{n+1}^3.$$

However, notice that since $x_1 < \ldots < x_n < x_{n+1}$,

$$x_1 + x_2 + \ldots + x_n \leq 1 + 2 + \ldots + (x_{n+1} - 1) = \frac{x_{n+1}(x_{n+1} - 1)}{2}.$$

For convenience, write $x_1 + x_2 + \ldots + x_n = c$ and $x_{n+1} = x$. Since $c \leq x(x-1)/2$, we have $x^2 \geq 2c + x$ and thus

$$x_1^3 + \cdots + x_{n+1}^3 \geq c^2 + x^3 \geq c^2 + x(2c + x) = (c + x)^2 = (x_1 + \ldots + x_{n+1})^2,$$

as desired.

5: **Answer.** $X - c$ for some integer c.

Suppose that f is monic of degree d. Clearly $d \neq 0$, and we are done if $d = 1$, so assume $d \geq 2$.

Pick a constant C such that f is strictly increasing on the interval (C, ∞) and $f(x) \leq f(C)$ for any $x \in [0, C]$. Choose some $N \geq 1$ such that $2^N \geq f(C)$. Then for $n \geq N$, the equation $f(a_n) = 2^n$ implies $a_n \geq C$. Since $f(a_m) > f(a_n)$ for $m > n$, we get $a_m > a_n$ for all $m > n \geq C$. In particular, we get $a_n \to \infty$ as $n \to \infty$.

Since f is monic of degree d, the quantity $f(x)^{1/d} - x$ converges as $x \to \infty$, say to $\beta \in \mathbb{R}$. Then we also have

$$\lim_{n\to\infty} 2^n - a_{nd} = \lim_{n\to\infty} f(a_{nd})^{1/d} - a_{nd} = \beta.$$

However, $2^n - a_{nd}$ is an integer for each n. Thus, β is an integer, and $2^n - a_{nd} = \beta \iff a_{nd} = 2^n - \beta$ for n large enough. We get $f(2^n - \beta) = 2^{nd}$ for n large enough, and so $f(x - \beta) - x^d$ is zero for infinitely many values of x. This gives us the polynomial equality $f(X) = (X + \beta)^d$.

Finally, the values attained by f are all dth powers. But $f(a_1) = 2$; contradiction.

6: Since $f \in \mathbb{Q}[x]$ has degree at least 2, the polynomial $f(x)^2 - (2x)^2$ has even degree with positive leading coefficient. Thus, there exists $M_1 > 0$ such that $|f(a)| > 2|a|$ for any $a \in \mathbb{Q}$ with $|a| > M_1$.

Next, for any $a \in \mathbb{Q}$, denote by $q(a)$ the smallest positive integer k such that ka is an integer. We claim that there exists $M_2 > 0$ such that $q(f(a)) > 2q(a)$. Let $n = \deg(f)$ and write

$$f(x) = \frac{1}{N}(a_0 x^n + a_1 x^{n-1} + \ldots + a_n),$$

where N is a positive integer and $a_0, a_1, \ldots, a_n$ are integers with $a_0 \neq 0$.

Fix some $a \in \mathbb{Q}$, and write $a = c/d$, where c and $d = q(a) > 0$ are coprime integers. Note that $q(f(a)) \cdot Nf(a)$ is an integer, and so

$$d^n \mid q(f(a)) \sum_{k=0}^{n} a_k c^{n-k} d^k.$$

Working mod d^2 yields

$$d^2 \mid q(f(a))(a_0 c^n + a_1 c^{n-1} d),$$

and thus

$$\frac{d^2}{\gcd(d^2, a_0c^n + a_1c^{n-1}d)} \mid q(f(a)).$$

We now bound $\gcd(d^2, a_0c^n + a_1c^{n-1}d)$. Since $\gcd(c, d) = 1$, we get

$$\gcd(d^2, a_0c^n + a_1c^{n-1}d) = \gcd(d^2, a_0c + a_1d).$$

Next, since $\gcd(c, d) = 1$, there exists an integer c_0 such that $cc_0 \equiv 1 \pmod{d}$. In particular, c_0 is invertible mod d^2, so Then we get

$$\begin{aligned}\gcd(d^2, a_0c + a_1d) &= \gcd(d^2, a_0c_0c + a_1c_0d)\\ &= \gcd(d^2, a_0 + a_1c_0d) \mid \gcd((a_1c_0d)^2, a_0 + a_1c_0d)\\ &= \gcd(a_0^2, a_0 + a_1c_0d) \leq a_0^2.\end{aligned}$$

As a result, we get $q(f(a)) \geq d^2/a_0^2 > 2d$ for $d = q(a) > 2|a_0|^2$. The second claim is proved by taking $M_2 = 2|a_0|^2$.

Now let $T = \{a \in \mathbb{Q} : |a| \leq M_1, q(a) \leq M_2\}$. There are finitely many choices of denominators for elements in T, and finitely many choices of numerators for each denominator, so T is finite. If $a \notin T$, then the choice of M_1 and M_2 yields either $|f(a)| > 2|a|$ or $q(f(a)) > 2q(a)$. By induction, for any k, either $|f^{[k]}(a)| > 2^k|a| > 2^kM_1$ or $q(f^{[k]}(a)) > 2^kq(a) > 2^kM_2$. We can now show that S is finite. In fact, we claim that $S \subseteq T$.

Fix some $a \in S$, and choose a sequence $(a_k)_{k\geq 1}$ of rational numbers such that $a = f^{[k]}(a_k)$ for all $k \geq 1$. The previous paragraph implies that $a_k \in T$ if $2^kM_1 \geq |a|$ and $2^kM_2 \geq q(a)$. Thus there exists N such that $a_k \in T$ for any $k \geq N$. By pigeonhole principle, there exists $i \geq N$ and $m > 0$ such that $b := a_i = a_{i+m}$. Then $a = f^{[i]}(b) = f^{[i+m]}(b)$. By induction, we get $f^{[n+m]}(b) = f^{[n]}(b)$ for all $n \geq i$. In particular, the orbit $\{f^{[k]}(b) : k \geq 0\} = \{f^{[k]}(b) : 0 \leq k < i + m\}$ of b is finite. Since a is in this orbit, the orbit $\{f^{[k]}(a) : k \geq 0\}$ of a is also finite. The previous paragraph then implies $a \in T$, as desired.