Week 9: Mock Putnam 3

1: Prove that there exist a sequence $n_1, n_2, \ldots$ of pairwise coprime integers such that $n_1 \cdots n_k - 1$ is the product of two consecutive integers for every positive integer k.

2: Let A be an $n \times n$ matrix with integer entries. Suppose p, q, r are positive integers such that $p^2 = q^2 + r^2$ and $p^2 A^{p^2} = q^2 A^{q^2} + r^2 I_n$, where I_n denotes the $n \times n$ identity matrix. Prove that $|\det A| = 1$.

3: Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ diverges. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + n a_n}$$

also diverges.

4: Let A be a 4×4 symmetric matrix with positive integer entries and $\det(A) = 1$. Prove that there exists a 4×4 matrix B with integer entries such that $A = B^t B$.

5: Let $f : [0, \infty) \to \mathbb{R}$ be twice differentiable with $f''(x) > 0$ for all $x \in [0, \infty)$ and $\lim_{x \to \infty} \frac{f(x)}{x} = \infty$. Prove that $\displaystyle\int_0^{\infty} \sin(f(x))\, dx$ converges, but not absolutely.

6: Let A be an infinite subset of the set of positive integers. Let x_n be the number of pairs $(a, b) \in A \times A$ such that $a < b$ and $a + b = n$. Prove that the sequence x_n is not eventually constant.

Week 9: Solutions

1: Prove that there exist a sequence $n_1, n_2, \ldots$ of pairwise coprime integers such that $n_1 \cdots n_k - 1$ is the product of two consecutive integers for every positive integer k.

We construct it inductively. Suppose $n_1 \cdots n_k - 1 = m(m+1)$ has been given. So $n_1 \cdots n_k = m^2 + m + 1$. Note that

$$(m^2 + m + 1)(m^2 - m + 1) = m^4 + m^2 + 1 = m^2(m^2 + 1) + 1.$$

Hence we take $n_{k+1} = m^2 - m + 1$. If q is a common divisor of $n_{k+1} = m^2 - m + 1$ and $n_1 \cdots n_k = m^2 + m + 1$, then q is odd and $q \mid 2m$. Hence $q \mid m$ but then $q \mid 1$. Hence n_{k+1} is coprime to $n_1 \cdots n_k$.

2: Let A be an $n \times n$ matrix with integer entries. Suppose p, q, r are positive integers such that $p^2 = q^2 + r^2$ and $p^2 A^{p^2} = q^2 A^{q^2} + r^2 I_n$, where I_n denotes the $n \times n$ identity matrix. Prove that $|\det A| = 1$.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then $p^2 \lambda^{p^2} = q^2 \lambda^{q^2} + r^2$. Note that $\lambda \neq 0$. We prove that $|\lambda| \leq 1$, which would imply that $|\det(A)|$ is a nonzero integer at most 1, hence equals 1. Suppose $|\lambda| > 1$. Then

$$|p^2 \lambda^{p^2}| = p^2 |\lambda|^{q^2} |\lambda|^{r^2} > (q^2 + r^2)|\lambda|^{q^2} > q^2 |\lambda|^{q^2} + r^2 \geq |q^2 \lambda^{q^2} + r^2|.$$

Contradiction.

3: Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ diverges. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + n a_n}$$

also diverges.

Let $b_n = \frac{a_n}{1 + n a_n}$. If $a_n < 1/n$, then $1 + n a_n < 2$ and we have $b_n > a_n/2$. If $a_n \geq 1/n$, then $1 + n a_n \geq 2$ and

$$b_n = \frac{1}{n} - \frac{1/n}{1 + n a_n} \geq \frac{1}{2n}.$$

Hence, it suffices to prove that the series

$$\sum_{n=1}^{\infty} \min\{a_n, \frac{1}{n}\}$$

diverges. Let $c_n = \min\{a_n, \frac{1}{n}\}$. Suppose for a contradiction that $\sum_{n=1}^{\infty} c_n$ converges. Suppose $\lim_{n \to \infty} n c_n \neq 0$. Then for some small $\delta > 0$, there exists arbitrarily large m such that $m c_m > \delta$. Then $m a_m > \delta$. Since a_n is decreasing, we see that for $m/2 \leq k \leq m$, we have $a_k > \delta/m$ and so $c_k > \min\{\delta/m, 1/m\} = \delta/m$ by choosing $\delta < 1$. Now

$$\sum_{m/2 \leq k \leq m} c_k > \frac{\delta}{2}.$$

Since there are infinitely such m, we see that $\sum_{k=1}^{\infty} c_k$ diverges. Contradiction. Hence we have $\lim_{n\to\infty} nc_n = 0$. This means that for n large enough, $c_n = a_n$ but the series $\sum a_n$ diverges.

4: Let A be a 4×4 symmetric matrix with positive integer entries and $\det(A) = 1$. Prove that there exists a 4×4 matrix B with integer entries such that $A = B^t B$.

We prove the general statement for all $n \times n$ positive definite matrices with $n \leq 4$. The statement is trivial for $n = 1$. Let $f_A(v) = v^t A v$ denote the associated quadratic form. Since the entries of A are positive, we see that f_A is positive definite. We first prove that there exists a nonzero $v_1 \in \mathbb{Z}^n$ such that $f_A(v_1) = 1$. The usual Gram-Schmidt gives a 4×4 matrix M with real coefficients such that $A = M^t M$. For any $w \in \mathbb{R}^n$, we have $f_A(w) = (Mw)^t(Mw)$. Hence the linear transformation M, which has determinant ± 1, is a map

$$\{w \in \mathbb{R}^n \colon f_A(w) < 2\} \to \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1^2 + \cdots + x_n^2 < 2\}.$$

Hence

$$\mathrm{Vol}\{w \in \mathbb{R}^n \colon f_A(w) < 2\} = \begin{cases} \frac{1}{2}\pi^2\sqrt{2}^4 & \text{if } n = 4, \\ \frac{4}{3}\pi\sqrt{2}^3 & \text{if } n = 3, \\ 2\pi\sqrt{2}^2 & \text{if } n = 2. \end{cases}$$

It is easy to check it is always bigger than 2^n. Hence by Minkowski's Theorem, there exists a nonzero lattice point $v_1 \in \mathbb{Z}^n$ such that $f_A(v) < 2$, which forces $f_A(v_1) = 1$.

We now mimic the Gram-Schmidt process. Denote $\langle x, y\rangle_A = x^t A y$. Since $f_A(v_1) = 1$, we see that no prime divides all the entries of v_1. Hence we can complete v_1 into a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{Z}^n$. For $i = 2, \ldots, n$, we let $w_i = v_i - \langle v_i, v_1\rangle_A v_1$ so that $\langle w_i, v_1\rangle_A = 0$. Let P be the $n \times n$ matrix with columns $w_1 = v_1, w_2, \ldots, w_n$. Then P has integer entries and $\det(P) = \pm 1$, implying that P^{-1} also has integer entries. Moreover,

$$P^t A P = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$$

where A' is a positive definite $(n-1)\times(n-1)$ matrix with $\det(A') = 1$. Applying induction A' gives an $(n-1)\times(n-1)$ matrix C with

$$P^t A P = \begin{pmatrix} 1 & 0 \\ 0 & C^t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}.$$

Hence $B = CP^{-1}$ does the job.

5: Let $f : [0,\infty) \to \mathbb{R}$ be twice differentiable with $f''(x) > 0$ for all $x \in [0,\infty)$ and $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$. Prove that $\displaystyle\int_0^{\infty} \sin(f(x))\, dx$ converges, but not absolutely.

First we note that $f'(x_0) > 0$ for some $x_0 \in [0,\infty)$ and so will be positive for all $x > x_0$. If $f'(x) < M$ for some positive M for all $x \in [0,\infty)$, then $f(x) < f(x_0) + M(x - x_0)$ and so $\lim_{x\to\infty} \frac{f(x)}{x} \leq M$. Hence we have $f'(x) \to \infty$ as $x \to \infty$. By shifting x, we may assume $f'(x) > 0$ for all $x \geq 0$. Now we apply integration by parts with $u = \frac{1}{f'(x)}$ and $dv = f'(x)\sin(f(x))dx$ to get

$$\int_a^b \sin(f(x))dx = -\frac{\cos(f(x))}{f'(x)}\bigg|_a^b - \int_a^b \cos(f(x))\frac{f''(x)}{(f'(x))^2}dx.$$

Note that

$$\left|\int_a^b \cos(f(x))\frac{f''(x)}{(f'(x))^2}dx\right| \le \int_a^b \frac{f''(x)}{(f'(x))^2}dx = \frac{1}{f'(a)} - \frac{1}{f'(b)}.$$

For $a, b > N$ where N is large enough, we have

$$\left|\int_a^b \cos(f(x))\frac{f''(x)}{(f'(x))^2}dx\right| \le \frac{1}{f'(N)} \to 0.$$

So the improper integral

$$\int_0^\infty \cos(f(x))\frac{f''(x)}{(f'(x))^2}dx$$

converges. Moreover,

$$\lim_{b\to\infty} -\left.\frac{\cos(f(x))}{f'(x)}\right|_0^b = \frac{1}{f'(0)}.$$

Hence the improper integral $\int_0^\infty \sin(f(x))dx$ converges.

For absolute convergence, for every integer $k > f(0)/\pi$, let $u_k \in [0, \infty)$ such that $f(u_k) = k\pi$. We break up the integral into the intervals $[u_k, u_{k+1}]$:

$$\int_{u_k}^{u_{k+1}} |\sin(f(x))|dx = \frac{1}{f'(u_k)} + \frac{1}{f'(u_{k+1})} \pm \int_{u_k}^{u_{k+1}} \cos(f(x))\frac{f''(x)}{(f'(x))^2}dx \ge \frac{2}{f'(u_{k+1})}.$$

It suffices to prove that

$$\sum_{k>f(0)/\pi+1} \frac{1}{f'(u_k)} = \infty.$$

Let $g(x) = f^{-1}(x)$ be the inverse of $f(x)$. Then $g'(x) = 1/f'(f^{-1}(x))$. So

$$\sum_{k>f(0)/\pi+1} \frac{1}{f'(u_k)} = \sum_{k>f(0)/\pi+1} g'(k\pi).$$

Since $f''(x) > 0$, we have $g''(x) < 0$. So the derivative $g'(x)$ is decreasing. Hence

$$\sum_{k>f(0)/\pi+1} g'(k\pi) \ge \int_{f(0)/\pi}^\infty g'(\pi x)\,dx = \frac{1}{\pi}\big(\lim_{u\to\infty} g(u) - g(f(0))\big) \to \infty.$$

6: Let A be an infinite subset of the set of positive integers. Let x_n be the number of pairs $(a, b) \in A\times A$ such that $a < b$ and $a + b = n$. Prove that the sequence x_n is not eventually constant.

Let $f(X) = \sum_{a\in A} X^a$. Then we see that

$$f(X)^2 - f(X^2) = \sum_{n=1}^\infty 2x_nX^n.$$

Suppose the sequence x_n is eventually constant. Then

$$f(X)^2 - f(X^2) = \frac{c}{1-X} + P(X)$$

for some integer c and polynomial $P(X) \in \mathbb{Z}[X]$. For X close to 1, we then have

$$f(X) \gg \frac{1}{\sqrt{1-X}}.$$

We now integrate

$$|f(X)|^2 \leq |f(X^2)| + \frac{c}{|1-X|} + |P(X)|$$

on the circle of radius r centered at the origin for $r < 1$ to get

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |f(r^2 e^{2i\theta})| \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} \frac{c}{|1-re^{i\theta}|} \, d\theta + O(1).$$

We can evaluate the LHS precisely via

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sum_{a,b\in A} r^{a+b} e^{i(a-b)\theta} \, d\theta = f(r^2)$$

using the fact that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\alpha\theta} \, d\theta = \begin{cases} 0 & \text{if } \alpha \neq 0 \\ 1 & \text{if } \alpha = 0. \end{cases}$$

By Cauchy-Schwartz, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(r^2 e^{2i\theta})| \, d\theta \leq \frac{1}{2\pi}\left(2\pi \int_0^{2\pi} |f(r^2 e^{2i\theta})|^2 \, d\theta\right)^{1/2} = \sqrt{f(r^4)} \leq \sqrt{f(r^2)}.$$

Hence, we have

$$f(r^2) \leq \sqrt{f(r^2)} + O\Big(\int_0^{2\pi} \frac{1}{|1-re^{i\theta}|} d\theta\Big) + O(1)$$

which implies that

$$\frac{1}{\sqrt{1-r}} \ll \frac{1}{\sqrt{1-r^2}} \ll f(r^2) \ll \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|} d\theta.$$

Note that

$$\frac{1}{|1-re^{i\theta}|} = \frac{1}{\sqrt{1-2r\cos\theta + r^2}} = \frac{1}{\sqrt{(1-r)^2 + 4r\sin^2(\theta/2)}}.$$

For $0 \leq \alpha \leq 1$, we have $\sin\alpha \geq \alpha - \alpha^3/6 \geq (5/6)\alpha$. Then

$$\begin{aligned} \int_0^{2\pi} \frac{1}{|1-re^{i\theta}|} d\theta &\leq 2\int_0^{\pi} \frac{1}{\sqrt{(1-r)^2 + (25/36)r\theta^2}} d\theta \\ &= 2 \cdot \frac{6}{5\sqrt{r}} \ln\left(\sqrt{(1-r)^2 + (25/36)r\theta^2} + \frac{5}{6}\sqrt{r}\theta\right)\Big|_0^{\pi} \\ &\ll \ln(1-r), \qquad \text{as } r \to 1^-. \end{aligned}$$

We now have a contradiction because

$$\lim_{r\to 1^-} \sqrt{1-r}\ln(1-r) = 0.$$