Week 8: Mock Putnam 2

1: Let p be a prime. Prove that for all $1 \leq k < p^2$, we have

$$\binom{2p^2}{k} \equiv 2\binom{p^2}{k} \pmod{p^2}.$$

2: Prove that the system

$$\begin{aligned} x^6 + x^3 + x^3y + y &= 69^{420} \\ x^3 + x^3y + y^2 + y + z^9 &= 420^{69} \end{aligned}$$

has no integer solutions in x, y, z.

3: Let $m > 1$ be an integer with at least two distinct prime divisors p and q. Prove that there does not exist a polynomial $f(x)$ with integer coefficients such that

- $f(n) \equiv 0 \pmod{m}$ for some integers n,
- $f(n) \equiv 1 \pmod{m}$ for some integers n,
- $f(n) \equiv 0$ or $1 \pmod{m}$ for all integers n.

4: Is it possible to partition the set of all nonnegative integers with at most 2023 digits into subsets of size 4 such that in each subset, the 4 numbers have the same digits in 2022 places and 4 consecutive digits in the remaining place?

5: Let $x_1, \ldots, x_5$ be nonnegative real numbers. Prove that

$$(x_1 + x_2 + \cdots + x_5)^3 \geq 25(x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2).$$

6: For any infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$, we write $f_n(x) = \frac{d}{dx} f_{n-1}(x)$ for $n \geq 1$ and $f_0(x) = f(x)$. Find all real numbers c such that there exists an differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) > f_{n-1}(x) + c$ for all positive integers n and all $x \in \mathbb{R}$.

Week 8: Solutions

1: Let p be a prime. Prove that for all $1 \le k < p^2$, we have

$$\binom{2p^2}{k} \equiv 2\binom{p^2}{k} \pmod{p^2}.$$

The left-hand-side is the coefficient of x^k in $(1+x)^{2p^2}$. Since $(1+x)^{p^2} \equiv 1 + x^{p^2} \pmod{p}$, there is a polynomial $A(x) \in \mathbb{Z}[x]$ such that

$$(1+x)^{p^2} = 1 + x^{p^2} + pA(x).$$

Squaring gives

$$(1+x)^{2p^2} \equiv 1 + 2pA(x) \pmod{x^{p^2}, p^2}.$$

The coefficient of x^k of $pA(x)$ is the same as the coefficient of x^k of $(1+x)^{p^2}$, which is $\binom{p^2}{k}$. Hence we have desired congruence mod p^2.

2: Prove that the system

$$\begin{aligned} x^6 + x^3 + x^3y + y &= 69^{420} \\ x^3 + x^3y + y^2 + y + z^9 &= 420^{69} \end{aligned}$$

has no integer solutions in x, y, z.

Adding the two equations gives

$$x^6 + 2x^3 + 2x^3y + 2y + y^2 + z^9 = (x^3 + y + 1)^2 - 1 + z^9 = 69^{420} + 420^{69}.$$

The z^9 suggests working mod 19 since it can only be $-1, 0, 1$ mod 19.

We have $69 \equiv -7 \pmod{19}$ so $69^2 \equiv 49 \equiv -8 \equiv (-2)^3 \pmod{19}$. Since $19 \equiv 3 \pmod{8}$, we know $-2 \equiv \alpha^2 \pmod{19}$ for some α and so $69^2 \equiv \alpha^6 \pmod{19}$. Hence $69^6 \equiv 1 \pmod{19}$ and so $69^{420} \equiv 1 \pmod{19}$.

We have $420 \equiv 2 \pmod{19}$ and $2^9 \equiv -1 \pmod{19}$. So $2^{69} \equiv -2^6 \equiv 12 \pmod{19}$. So

$$(x^3 + y + 1)^2 \equiv 69^{420} + 420^{69} + 1 - z^9 \equiv 13, 14 \text{ or } 15 \pmod{19}.$$

We now compute Legendre symbols to show $13, 14, 15$ are all quadratic nonresidues mod 19:

$$\left(\frac{13}{19}\right) = \left(\frac{19}{13}\right) = \left(\frac{6}{13}\right) = \left(\frac{2}{13}\right)\left(\frac{3}{13}\right) = -1,$$

since $13 \not\equiv \pm 1 \pmod{8}$ and $13 \equiv \pm 1 \pmod{12}$;

$$\left(\frac{14}{19}\right) = \left(\frac{2}{19}\right)\left(\frac{-12}{19}\right) = \left(\frac{2}{19}\right)\left(\frac{-1}{19}\right)\left(\frac{3}{19}\right) = -1.$$

since $19 \not\equiv \pm 1 \pmod 9$ and $19 \not\equiv 1 \pmod 4$ and $19 \not\equiv \pm 1 \pmod{12}$;

$$\left(\frac{15}{19}\right) = \left(\frac{-4}{19}\right) = \left(\frac{-1}{19}\right) = -1.$$

3: Let $m > 1$ be an integer with at least two distinct prime divisors p and q. Prove that there does not exist a polynomial $f(x)$ with integer coefficients such that

- $f(n) \equiv 0 \pmod m$ for some integers n,
- $f(n) \equiv 1 \pmod m$ for some integers n,
- $f(n) \equiv 0$ or $1 \pmod m$ for all integers n.

Let $a \in \mathbb{Z}$ such that $f(a) \equiv 0 \pmod m$. Fix any integer k. We have

$$f(a+kp) \equiv f(a) \pmod p$$

and so we can't have $f(a+kp) \equiv 1 \pmod m$ as that would imply $p \mid 1$. Hence $f(a+kp) \equiv 0 \pmod m$. Now fix any integer ℓ. We have

$$f(a+kp+\ell q) \equiv f(a+kp) \pmod q$$

and so we can't have $f(a+kp+\ell q) \equiv 1 \pmod m$ as that would imply $q \mid 1$. Hence $f(a+kp+\ell q) \equiv 0 \pmod m$. However, since p, q are distinct primes, every integer is of the form $a + kp + \ell q$ for some $k, \ell \in \mathbb{Z}$. This contradicts the assumption that $f(n) \equiv 1 \pmod m$ for some integer n.

4: Is it possible to partition the set of all nonnegative integers with at most 2023 digits into subsets of size 4 such that in each subset, the 4 numbers have the same digits in 2022 places and 4 consecutive digits in the remaining place?

Let S be the set of all positive integers with at most 2023 digits. For any $n \in S$, let $s(n)$ denote the sum of the digits of n. Consider the generating function

$$F(X) = \sum_{n \in S} X^{s(n)} = (1 + X + \cdots + X^9)^{2023}.$$

On the other hand, if T is a set of 4 numbers that have the same digits in 2022 places and 4 consecutive digits in the remaining place, then their digit sums are 4 consecutive integers and so

$$1 + X + X^2 + X^3 \mid \sum_{n \in T} X^{s(n)}.$$

Hence, if such a decomposition is possible, then

$$1 + X + X^2 + X^3 \mid (X^{10} - 1)^{2023}$$

which is not possible since i is not a root of $X^{10} - 1$ but is a root of $1 + X + X^2 + X^3$.

5: Let $x_1, \ldots, x_5$ be nonnegative real numbers. Prove that

$$(x_1 + x_2 + \cdots + x_5)^3 \geq 25(x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2).$$

By cyclicity, we may assume x_5 is the smallest. We have $x_1 + x_4 - x_5 \geq 0$ and so

$$\begin{aligned} \text{RHS} &= x_5(x_1+x_3)(x_2+x_4) + x_2x_3(x_1+x_4-x_5) \\ &\leq \frac{1}{4}x_5(x_1+x_2+x_3+x_4)^2 + \frac{1}{27}(x_1+x_2+x_3+x_4-x_5)^3. \end{aligned}$$

If $x_5 = 0$, then it is less than or equal to $(x_1+x_2+x_3+x_4)^3/25$ as desired. Suppose now $x_5 \neq 0$. By scaling, we may assume $x_5 = 1$. Let $t = x_1 + x_2 + x_3 + x_4$. It remains to prove that

$$\frac{1}{25}(t+1)^3 - \frac{1}{4}t^2 - \frac{1}{27}(t-1)^3 \geq 0$$

for $t \geq 4$. It takes the value 0 when $t = 4$ and has derivative

$$\begin{aligned} \frac{3}{25}(t+1)^2 - \frac{1}{2}t - \frac{1}{9}(t-1)^2 &= \frac{1}{450}(54t^2 + 108t + 54 - 225t - 50t^2 + 100t - 50) \\ &= \frac{1}{450}(4t^2 - 17t + 4) \\ &= \frac{1}{450}(4t-1)(t-4) \end{aligned}$$

is positive for $t > 4$. Hence the cubic above is non-negative for $t \geq 4$.

6: For any infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$, we write $f_n(x) = \frac{d}{dx}f_{n-1}(x)$ for $n \geq 1$ and $f_0(x) = f(x)$. Find all real numbers c such that there exists an differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) > f_{n-1}(x) + c$ for all positive integers n and all $x \in \mathbb{R}$.

By taking $f(x) = e^{2x}$, we see that $f_n(x) = 2^n e^{2x} > 2^{n-1}e^{2x}$ for all $x \in \mathbb{R}$. Hence every $c \leq 0$ works. Suppose $c > 0$ and such a function $f(x)$ exists.

Solution 1: Let $g(x) = f'(x) - f(x)$. Then $g(x) > c$ and $g'(x) > c$ for all $x \in \mathbb{R}$. The condition $g'(x) > c$ implies that $\lim_{x\to-\infty} g(x) = -\infty$, which contradicts $g(x) > c$ for all x.

Solution 2: Note that for any $a \in \mathbb{R}$, if $f(a) > -c/2$, then $f'(a) > c/2$. This implies that for any $b \in \mathbb{R}$, if $f(a) > -c/2$ for all $a \leq b$, then $f'(a) > c/2$ for all $c \leq b$ and so $\lim_{x\to-\infty} f(x) = -\infty$, which is a contradiction. Hence, for any $b \in \mathbb{R}$, there exists some $a \leq b$ such that $f(a) \leq -c/2$.

Moreover, since the function is strictly increasing whenever $f(x) > -c/2$, we see that if $f(b) > -c/2$ for some $b \in \mathbb{R}$, then $f(x) > -c/2$ for all $x > b$. To prove this rigorously, suppose $f(a) \leq -c/2$ for some $a > b$. Let $a_0 = \inf\{a > b\colon f(a) \leq -c/2\}$. By continuity, $f(a_0) \leq -c/2$. By MVT, there exists some $a_1 \in (b, a_0)$ such that $f(a_0) - f(b) = f'(a_1)(b - a_0)$, but $f(a_0) - f(b) < 0$ and $f'(a_1)(b - a_0) > 0$. Note this also implies that if $f(b) \leq -c/2$ for some $b \in \mathbb{R}$, then $f(x) \leq -c/2$ for all $x \leq b$.

From the above two paragraphs, we have some $b_0 \in \mathbb{R}$ such that $f(x) \leq -c/2$ for all $x \leq b_0$. The trick now is to note that $f'(x)$ also satisfies the given condition. Hence applying the above to $f'(x)$, we see that there exists some $b_1 \leq b_0$ such that $f'(x) \leq -c/2$ for all $x \leq b_1$. This implies that $\lim_{x\to-\infty} f(x) = \infty$ which contradicts $f(x) \leq -c/2$ for all $x \leq b_1$.