Week 7: Mock Putnam 1

1: Suppose $f : [0, 1] \to \mathbb{R}$ is continuous differentiable with $f(0) = 0$ and $f(1) = 1$. Prove that for any $n \in \mathbb{N}$, there exists distinct $x_1, \ldots, x_n \in (0, 1)$ such that $\displaystyle\sum_{i=1}^{n} \frac{1}{(f'(x_i))^{2023}} = n$.

2: Prove that for any positive integer n,

$$(n + 1) \operatorname{lcm}\left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right) = \operatorname{lcm}(1, 2, \ldots, n + 1).$$

3: Let $A \subseteq \mathbb{Z}/2023\mathbb{Z}$ be a subset with 289 elements. Prove that there exists a subset $B \subseteq \mathbb{Z}/2023\mathbb{Z}$ with 5 elements such that

$$|A + B| = |\{a + b \colon a \in A, b \in B\} \geq 1000.$$

4: Let $P(x)$ be a polynomial of degree at most 6^{337} with coefficients in the set $\{-1, 0, 1\}$. Suppose that $(x - 1)^{2023} \mid P(x)$ in $\mathbb{Z}[x]$. Prove that the polynomial $P(x)$ vanishes at all primitive 7-th roots of unities.

5: Suppose $f(x)$ is a polynomial with integer coefficients such that

$$f(0), f(1), \ldots, f(2024) \in \{0, 1, \ldots, 2023\}.$$

Prove that $f(0) = f(1) = \cdots = f(2024)$.

6: Let S be the set of all continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$\left| \int_0^x \frac{f(t)}{\sqrt{x - t}}\, dt \right| \leq 1, \qquad \text{for all } x \in [0, 1].$$

Find $\displaystyle\sup_{f \in S} \int_0^1 f(x)\, dx$.

Week 7: Solutions

1: Suppose $f : [0,1] \to \mathbb{R}$ is continuous differentiable with $f(0) = 0$ and $f(1) = 1$. Prove that for any $n \in \mathbb{N}$, there exists distinct $x_1, \ldots, x_n \in (0,1)$ such that $\displaystyle\sum_{i=1}^{n} \frac{1}{(f'(x_i))^{2023}} = n$.

By the Mean Value Theorem, there exists some $c_1 \in (0,1)$ such that $f'(c_1) = 1$. If $f(c_1) = c_1$, then we can find $c_2 \in (0, c_1)$ such that $f'(c_2) = 1$. If this process continues to find $c_1 > c_2 > \cdots > c_n$ such that $f(c_i) = c_i$ and $f'(c_i) = 1$ for $i = 1, \ldots, n$, we are done by taking $x_i = c_i$. Otherwise, let k be the smallest positive integer such that $f(c_k) \neq c_k$. Since $f(0) = 0$ and $f(c_{k-1}) = c_{k-1}$, where we set $c_0 = 1$, we can find $y, z \in (0,1)$ such that $f'(y) > 1$ and $f'(z) < 1$. Since $f'(x)$ is continuous, we see that $f'(x)$ takes every value in some open interval I containing 1 as x varies between y and z. There then exist an increasing sequence (a_m) in $I \cap (0,1)$ and an decreasing sequence (b_m) in $I \cap (1, \infty)$ such that

$$\frac{1}{a_m^{2023}} + \frac{1}{b_m^{2023}} = 2, \quad \text{for all } m \geq 1.$$

Let $y_m, z_m \in (0,1)$ such that $f'(y_m) = a_m$ and $f'(z_m) = b_m$. If $n = 2\ell$ is even, we take $x_1, \ldots, x_n = y_1, z_1 \ldots, y_\ell, z_\ell$. If $n = 2\ell + 1$, we add $x_1, \ldots, x_n = c_1, y_1, z_1 \ldots, y_\ell, z_\ell$.

2: Prove that for any positive integer n,

$$(n+1)\,\mathrm{lcm}\left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right) = \mathrm{lcm}(1, 2, \ldots, n+1).$$

We note that for $i \leq n$,

$$(n+1)\binom{n}{i} = \frac{(n+1)!}{i!(n-i)!} = (i+1)\binom{n+1}{i+1}.$$

Let $p \leq n+1$ be any prime and let k be the unique positive integer such that $p^k \leq n+1 < p^{k+1}$. Then $\nu_p(\mathrm{lcm}(1, 2, \ldots, n+1)) = k$. Let $i = p^k - 1 \leq n$. Then

$$\nu_p\left((i+1)\binom{n+1}{i+1}\right) \geq \nu_p(i+1) = k.$$

Next fix any $m = 1, \ldots, n+1$. Suppose $\ell = \nu_p(m) \leq k$. Then the remainders when m is divided by $p, p^2, \ldots, p^\ell$ are all 0. Hence, we have

$$\left\lfloor \frac{n+1}{p^j} \right\rfloor - \left\lfloor \frac{m}{p^j} \right\rfloor - \left\lfloor \frac{n+1-m}{p^j} \right\rfloor = 0 \qquad \text{if } j \leq \ell;$$

is at most 1 for $j = \ell + 1, \ldots, k$; and is 0 for $j \geq k+1$. Hence

$$\nu_p\left(m\binom{n+1}{m}\right) \leq \ell + (k - \ell) = k.$$

This proves that the two sides have the same p-adic valuation for every prime p that appears. Hence they are equal.

3: Let $A \subseteq \mathbb{Z}/2023\mathbb{Z}$ be a subset with 289 elements. Prove that there exists a subset $B \subseteq \mathbb{Z}/2023\mathbb{Z}$ with 5 elements such that

$$|A+B| = |\{a+b\colon a \in A, b \in B\} \geq 1000.$$

Let S be the set of all 5-elements subsets of $\mathbb{Z}/2023\mathbb{Z}$. Then

$$\sum_{B\in S} |A+B| = \sum_{n=0}^{2023} |\{B \in S\colon n \in A+B\}| = \sum_{n=0}^{2023} \binom{2023}{5} - \binom{2023-289}{5}.$$

Hence, it suffices to prove that

$$2023\binom{2023-289}{5} \leq (2023-1000)\binom{2023}{5},$$

which follows from

$$\frac{\binom{2023-289}{5}}{\binom{2023}{5}} = \prod_{k=0}^{4} \frac{2023-289-k}{2023-k} \leq \left(\frac{2023-289}{2023}\right)^5 = \left(\frac{6}{7}\right)^5 = \frac{7776}{16807} \leq \frac{1}{2} \leq \frac{1013}{2023}.$$

4: Let $P(x)$ be a polynomial of degree at most 6^{337} with coefficients in the set $\{-1, 0, 1\}$. Suppose that $(x-1)^{2023} \mid P(x)$ in $\mathbb{Z}[x]$. Prove that the polynomial $P(x)$ vanishes at all primitive 7-th roots of unities.

Let $Q(x) \in \mathbb{Z}[x]$ such that $P(x) = (x-1)^{2023}Q(x)$. Suppose $P(\zeta_7) \neq 0$. Then $Q(\zeta_7) \neq 0$. Since $Q(x) \in \mathbb{Q}[x]$, we know that $Q(\zeta_7^j) \neq 0$ for $j = 1, \ldots, 6$. Hence

$$\left|\prod_{j=1}^{6} Q(\zeta_7^j)\right| \geq 1$$

since it is a nonzero integer. Moreover,

$$\prod_{j=1}^{6} (\zeta_7^j - 1) = \Phi_7(1) = 7$$

where $\Phi_7(x) = x^6 + x^5 + \cdots + 1$ is the 7-th cyclotomic polynomial. Hence

$$\left|\prod_{j=1}^{6} P(\zeta_7^j)\right| = \left|\prod_{j=1}^{6} (\zeta_7^j - 1)\right|^{2023} \left|\prod_{j=1}^{6} Q(\zeta_7^j)\right| \geq 7^{2023}.$$

On the other hand, since the coefficients of $P(x)$ have absolute value at 1 and $|\zeta_7^j| = 1$, we have by triangle inequality that

$$|P(\zeta_7^j)| \leq 6^{337} + 1 \leq 7^{337}.$$

Hence
$$\Big|\prod_{j=1}^{6} P(\zeta_7^j)\Big| \le (7^{337})^6 = 7^{2022} < 7^{2023}.$$
Contradiction.

5: Suppose $f(x)$ is a polynomial with integer coefficients such that
$$f(0), f(1), \dots, f(2024) \in \{0, 1, \dots, 2023\}.$$
Prove that $f(0) = f(1) = \cdots = f(2024)$.

Since $f(0) \equiv f(2024) \pmod{2024}$, we see that $f(0) = f(2024)$. Let $g(x) = f(x) - f(0)$. Then
$$g(x) = x(x - 2024)h(x)$$
for some $h(x) \in \mathbb{Z}[x]$. For $x = 2, \dots, 2022$, we have $|x(x - 2024)| > 2023$ while $|g(x)| \le 2023$, implying that $h(x) = 0$. So
$$g(x) = x(x-2)(x-3)\cdots(x-2021)(x-2022)(x-2024)j(x)$$
for some $h(x) \in \mathbb{Z}[x]$. We can now set $x = 1$ or $x = 2023$ like above to get $j(x) = 0$ in these cases. So
$$f(x) - f(0) = x(x-1)\cdots(x-2023)(x-2024)k(x)$$
for some $k(x) \in \mathbb{Z}[x]$, which implies that $f(0) = f(1) = \cdots = f(2024)$.

6: Let S be the set of all continuous function $f : [0, 1] \to \mathbb{R}$ such that
$$\left|\int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt\right| \le 1, \qquad \text{for all } x \in [0, 1].$$
Find $\displaystyle\sup_{f\in S} \int_0^1 f(x)\,dx$.

The key trick is that the following integral
$$\int_t^u \frac{1}{\sqrt{(x-t)(u-x)}}\,dx$$
where $t < u$ does not depend on t and u, by taking $v = u - x$ and then $w = v/(u - t)$. By taling $t = -1$ and $u = 1$, we have
$$\int_t^u \frac{1}{\sqrt{(x-t)(u-x)}}\,dx = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x\big|_{-1}^1 = \pi.$$
In particular, we have
$$\int_t^1 \frac{1}{\sqrt{(x-t)(1-x)}}\,dx = \int_0^x \frac{1}{\sqrt{(x-t)(t-0)}}\,dt = \pi.$$

Returning to the problem, we find that

$$\begin{aligned}\int_0^1 \left(\int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt\right)\frac{1}{\sqrt{1-x}}\,dx &= \int_0^1 \left(\int_t^1 \frac{1}{\sqrt{(x-t)(1-x)}}\,dx\right) f(t)\,dt \\ &= \pi\int_0^1 f(t)\,dt.\end{aligned}$$

Hence

$$\int_0^1 f(t)\,dt \le \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \frac{2}{\pi}.$$

Equality is achieved when $f(x) = \frac{1}{\pi\sqrt{x}}$ in which case

$$\int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt = \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{(x-t)(t-0)}}\,dt = 1.$$

There is one slight subtlety here as the function $\frac{1}{\pi\sqrt{x}}$ is not continuous on $[0,1]$. For any positive integer n, we let

$$f_n(x) = \frac{1}{\pi\sqrt{x+1/n}}.$$

Then we have

$$0 \le f_n(x) \le \frac{1}{\pi\sqrt{x}}, \qquad \text{and so} \qquad f_n(x)\in S.$$

Since $f_n(x) \to \frac{1}{\pi\sqrt{x}}$ pointwise, we have by Lebesgue Dominated Convergence Theorem,

$$\int_0^1 f_n(x)\,dx \to \int_0^1 \frac{1}{\pi\sqrt{x}}\,dx = \frac{2}{\pi}.$$

Therefore, $2/\pi$ is the desired supremum.