Week 6: Assorted Problems

1: Prove Cauchy's mean value theorem: Let $f, g : [a, b] \to \infty$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$.

2: Let $a_1 = 2$ and $a_n = 2^{a_{n-1}}$ for $n \geq 2$. Find the smallest n such that $a_n \geq 16^{16^{16^{16}}}$.

3: Prove that every positive rational number can be expressed as a quotient of products of numbers of the form $\binom{n}{\lfloor n/2 \rfloor}$ where n is a positive integer.

4: Let $g(z) : \mathbb{C} \to \mathbb{C}$ be an entire (complex analytic on $\mathbb{C}$) function such that $g(z^2) = g(z) + g(z-1)$ for all $z \in \mathbb{C}$. Prove that $g(z) = 0$.

5: Let $a_0 = 2$. For $n \geq 1$, let a_n be the smallest positive integer such that $\sum_{j=0}^{n} \frac{1}{a_j} < 1$. Prove that $\sum_{n=1}^{\infty} \frac{1}{\log_2(a_n)}$ converges.

6: Let p be a prime and let G be a subgroup of $\mathbb{F}_p^\times$ of order divisible by 6. Prove that there exist $a, b, c \in G$ such that $a + b = c$.

7: Let $n \geq 1$ and let $a_1 \leq a_2 \leq \cdots \leq a_n$ be real numbers such that $a_1 + 2a_2 + \cdots + na_n = 0$. Prove that for any real number x, we have $\sum_{i=1}^{n} a_i \lfloor ix \rfloor \geq 0$.

8: Let $f : [0, \infty) \to \mathbb{R}$ be differentiable with $|f(x)|$ bounded and $f(x)f'(x) \geq \cos x$ for all x. Prove that $\lim_{x\to\infty} f(x)$ does not exist.

9: Find all positive integers m such that for $n = 4m(2^m - 1)$, we have $n \mid a^m - 1$ for all a coprime to n.

10: Use the Polya-Vinogradov inequality $\left|\sum_{a=1}^{m} \left(\frac{a}{p}\right)\right| \leq \sqrt{p}\log p$ to prove that there exists $1 \leq a < p^{\frac{1}{2\sqrt{e}}}(\log p)^2$ that is a quadratic non-residue mod p.

11: Suppose $f : (0, \infty) \to \mathbb{R}$ is differentiable with $\lim_{x\to\infty} \left(f(x) + \frac{f'(x)}{x}\right) = 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.

12: Prove Sperner's Theorem. Let S be a set of subsets of $\{1, 2, \ldots, n\}$ such that for any $A, B \in S$, either $A \not\subseteq B$ or $B \not\subseteq A$. Then $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Week 6: Hints

1: Let $h(x) = f(x) - rg(x)$ where $r \in \mathbb{R}$ is chosen so that $h(a) = h(b)$.

2: Take $\log_2$ a few times.

3: It is enough to consider only n prime.

4: Prove that there exists some $M > 0$ and $d > 0$ such that $|g(z)| \leq M|z|^d$. This implies that g is a polynomial.

5: Prove that $a_{n+1} = a_n^2 - a_n + 1$.

6: G is cyclic. Let $\alpha \in G$ be an element of order 6.

7: Induct on n. Let $b_i = a_i + (2/n)a_{n+1}$.

8: Consider $F(x) = f(x)^2 - \sin x$.

9: Move to the sphere. Prove that the surface area of the part of the sphere of radius R bounded within two parallel planes of distance d is equal to $2\pi Rd$.

10: Let N be the number of quadratic non-residues among $1, 2, \ldots, m$. Let X be the smallest quadratic non-residue. Then every quadratic non-residue has a prime divisor at least X. So $N \leq \sum_{X \leq p < m} \frac{m}{p}$.

Polya-Vinogradov gives $|m - 2N| \leq \sqrt{p} \log p$.

11: Apply Cauchy's mean value theorem to $g(x) = e^{x^2/2} f(x)$ and $h(x) = e^{x^2/2}$.

12: For any permutation σ of $\{1, 2, \ldots, n\}$ and any $i = 1, \ldots, n$, let $A_i(\sigma)$ be 1 if $\{\sigma(1), \ldots, \sigma(i)\} \in S$ and 0 otherwise. Let $X(\sigma) = \sum_{i=1}^n A_i(\sigma)$. Then $X(\sigma) \leq 1$ for all σ. Compute $\sum_\sigma X(\sigma)$ via $\sum_\sigma A_i(\sigma)$.