Week 5: Assorted Problems

1: Find all integer solutions to $a^2+b^2+c^2=a^2b^2$.

2: For any $m \in \mathbb{N}$, let $\Phi_m(x)$ denote the m-th cyclotomic polynomial. Prove that $\Phi_{420}(69) > \Phi_{69}(420)$.

3: Let $c \in \mathbb{R}$ such that the polynomial $4x^3-3x+c=0$ has a root in $[-1,1]$. Prove that it has all of its roots in $[-1,1]$.

4: Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)=2f(x)f(y)$ for all $x,y\in\mathbb{R}$.

5: Let A,B be $n\times n$ matrices with coefficients in $\mathbb{R}$ such that $AB-BA=A$. Prove that A is nilpotent. That is, $A^k=0$ for some $k\in\mathbb{N}$.

6: Let G be a group and let H be a subgroup such that for any $a\in G\backslash H$ and any $g\in G$, there exists $b\in H$ such that $gag^{-1}=bab^{-1}$. Prove that H is a normal subgroup of G and G/H is abelian.

7: Let $v_1,\dots,v_k$ be nonzero vectors in $\mathbb{R}^n$ such that $v_i^t v_j \le 0$ for all $i\ne j$. Prove that $k\le 2n$.

8: Let n be a positive integer such that there exists a polynomial $f(x)\in\mathbb{F}_2[x]$ of degree n such that $f(x)\cdot x^n f(1/x)=1+x+\cdots+x^{2n}$ in $\mathbb{F}_2[x]$. Prove that the order of 2 mod $2n+1$ (the smallest positive integer d such that $2^d\equiv 1 \pmod{2n+1}$) is odd.

9: Let $\alpha,\beta,\gamma\in\mathbb{C}$ be the three roots of x^3+x+1. For any $n\in\mathbb{N}$, let $a_n=\dfrac{(\alpha^n-1)(\beta^n-1)(\gamma^n-1)}{(\alpha-1)(\beta-1)(\gamma-1)}$. Prove that $a_n\ge 1$ for all $n\ge 1$ with equality exactly when $n=1,2,4,5$.

10: Let p be a prime. Let A and B be two square matrices with complex coefficients such that $AB=BA$ and $A^{p-1}=B^p=I$, the identity matrix. Prove that $A+B+I$ is invertible.

11: Let $f:(1,\infty)\to\mathbb{R}$ be a continuously differentiable function such that $f(x)\le x^2\log x$ and $f'(x)>0$ for all $x>1$. Prove that $\displaystyle\int_1^\infty \frac{1}{f'(x)}\,dx=\infty$.

12: Let $k\in\mathbb{N}$ such that $p=6k+1$ is prime. Prove that $\dbinom{3k}{k}\not\equiv 1 \pmod{p}$.

Week 5: Hints

1: Work mod 4.

2: $\Phi_m(x)$ is a product of $\phi(m)$ terms of the form $x - \zeta$ where $|\zeta| = 1$.

3: $4\cos^3\alpha - 3\cos\alpha = \cos(3\alpha)$.

4: By scaling, we can remove 2.

5: $A^kB - BA^k = kA^k$. The linear operator $L(A) = AB - BA$ has too many distinct eigenvalues.

6: For $u \in H$ and $g \in G$, apply the assumption to $a = gug^{-1}$ if it does not belong to H,

7: Induct on n.

8: The existence of such a polynomial $f(x)$ is equivalent to requiring that for any irreducible factor of $x^{2n+1} - 1$ besides $x - 1$, if α is a root, then $1/\alpha$ is not a root. What do roots of an irreducible polynomial in $\mathbb{F}_p[x]$ look like?

9: Note that a_n is a linear combination of 7 n-th powers. Write down a recursion formula satisfied by a_n.

10: Suppose $(A + B + I)v = 0$. Prove that $f(B)v = g(B)v = 0$ where $f(x) = (x+1)^{p-1} - 1$ and $g(x) = x^p - 1$.

11: Use Cauchy-Schwartz on $\int_1^\infty \frac{1}{f'(x)}\,dx \cdot \int_1^\infty \frac{f'(x)}{(x\log x)^2}\,dx$. Estimate the second integral by integration by parts.

12: Let $a \in \mathbb{F}_p$ be an element of order k. Consider $\sum_{j=1}^{k}(1 + a^j)^{3k}$.