Week 4: Assorted Problems

1: Let a, b, c be real numbers such that $a+b+c=0$. Prove that

$$\frac{a^5+b^5+c^5}{5} = \frac{a^3+b^3+c^3}{3}\frac{a^2+b^2+c^2}{2}.$$

2: Prove that 2 is a primitive root mod 101. In other words, the smallest positive integer d such that $2^d \equiv 1 \pmod{101}$ is 100.

3: Let a_{ij} be positive real numbers. Prove that

$$\prod_{i=1}^{k}(a_{i1}+a_{i2}+\cdots+a_{in}) \geq \left(\sqrt[k]{a_{11}a_{21}\cdots a_{k1}}+\cdots+\sqrt[k]{a_{1n}a_{2n}\cdots a_{kn}}\right)^k.$$

4: Let $f(x) \in \mathbb{R}[x]$ be a quadratic polynomial with $|f(-1)| \leq 1, |f(0)| \leq 1, |f(1)| \leq 1$. Prove that for any $|x| \leq 1$, we have $|f(x)| \leq 5/4$.

5: Let $S \subseteq \mathbb{R}$ be a subset such that any finite sum of elements in S has absolute value at most 1. Prove that S is countable.

6: Let $D_1, \ldots, D_n$ be n disks in $\mathbb{R}^2$. For any i, j, let a_{ij} denote the area of $D_i \cap D_j$. Prove that for any real numbers $x_1, \ldots, x_n$, we have $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j \geq 0$.

7: Let $n \geq 2$ be an integer. Let $a = n + \sqrt{n^2 - n}$. Prove that for any positive integer m, we have $\lfloor a^m \rfloor \equiv -1 \pmod{n}$. Does there exist an irrational number b such that $\lfloor b^m \rfloor \equiv 0 \pmod{n}$ for any positive integer m?

8: Find the limit $\displaystyle\lim_{n\to\infty}\sqrt{1+2\sqrt{1+3\sqrt{\cdots+(n-1)\sqrt{1+n}}}}$.

9: Suppose a, b are positive rational numbers and $n \geq 2$ is an integer such that $\sqrt[n]{a} + \sqrt[n]{b} \in \mathbb{Q}$. Prove that $\sqrt[n]{a} \in \mathbb{Q}$.

10: Consider the sequence defined by $a_0 = a_1 = 1$ and $a_{n+2} = a_{n+1} + \displaystyle\sum_{k=0}^{n} a_k a_{n-k}$ for $n \geq 0$. Prove that

$$a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{k+1}\binom{2k}{k}\binom{n}{2k}.$$

11: Let $f : (0, \infty) \to \mathbb{R}$ be a twice continuously differentiable function such that

$$|f''(x) + 2xf'(x) + (x^2 + 1)f(x)|$$

is bounded. Prove that $\lim_{x \to \infty} f(x) = 0$.

12: Prove that there do not exist 22 (distinct) circles $C_1, \dots, C_{22}$ in $\mathbb{R}^2$ and 22 (distinct) points $P_1, \dots, P_{22}$ on $C_1 \cup \dots \cup C_{22}$ such that every circle C_i contains at least 7 points in $P_1, \dots, P_{22}$ and every point P_i lies on at least 7 circles $C_1, \dots, C_{22}$.

Week 4: Hints

1: Bash.

2: Note that $a^{(p-1)/2} = \left(\frac{a}{p}\right)$.

3: Let $S_i = a_{i1} + \cdots + a_{in}$. Apply AM-GM to $\sqrt[k]{\dfrac{a_{1j}a_{2j}\cdots a_{kj}}{S_1 S_2 \cdots S_k}}$ and sum over j.

4: Lagrange interpolation.

5: Consider sets of the form $S \cap (1/n, \infty)$.

6: Let χ_{D_i} denote the characteristic function of D_i. Then $a_{ij} = \displaystyle\int_{\mathbb{R}^2} \chi_{D_i}(x,y)\chi_{D_j}(x,y)dxdy$.

7: Suppose $(x-a)(x-c) = x^2 + ux + v \in \mathbb{Z}[x]$ with $0 < c < 1$. Then $\lfloor a^m \rfloor = a^m + c^m - 1$. Use a recurrence relation between $a^{m+2} + c^{m+2}$, $a^{m+1} + c^{m+1}$ and $a^m + c^m$ to find it mod n.

8: Let $a_{m,n} = \sqrt{1 + m\sqrt{1 + (m+1)\sqrt{\cdots + (n-1)\sqrt{1+n}}}}$. Consider $|a_{m,n} - (m+1)|$.

9: Let $c = \sqrt[n]{a} + \sqrt[n]{b}$. Prove that $x^n - a$ and $(c-x)^n - b$ share a unique common root (in $\mathbb{C}$).

10: Generating functions.

11: Consider $f(x)e^{x^2/2}$.

12: Consider the incidence correspondence $(P, (C_i, C_j))$ where $P \in C_i \cap C_j$. Then consider the matrix A whose (i,j)-entry is 1 if $P_i \in C_j$ and 0 otherwise. What is A^tA and what is its determinant?