Week 3: Assorted Problems

1: Let $a > 1$ be an integer. Prove that there exist positive integers b, c such that $a^2 + b^2 = c^2$.

2: Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of real numbers such that $b_n \neq 0$ and $a_n + b_n \neq 0$ for all $n \in \mathbb{N}$. Suppose the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{b_n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{a_n^2}{b_n^2}$ converge. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{a_n + b_n}$ converges.

3: Let $A = \{(x, y) \in \mathbb{R}^2 \colon x = 0 \text{ or } y = 0\}$ equipped with the subspace topology from $\mathbb{R}^2$. Let $f : \mathbb{R} \to A$ be a surjective and continuous map. Prove that $f^{-1}((0,0))$ is infinite.

4: Let A be a 3×2 matrix and let B be a 2×3 matrix with complex coefficients such that

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

Find BA.

5: For any subset $S \subseteq \mathbb{N}$, we write $S \oplus S = \{a + b \colon a, b \in S, a \neq b\}$. Prove that there exist a unique partition of $\mathbb{N}$ into (disjoint subsets) A and B such that neither of $A \oplus A$ and $B \oplus B$ contains a prime.

6: Prove that if $f : \mathbb{N} \to \mathbb{N}$ is a function satisfying $af(b) + b(a) = (a + b)f(ab + a + b)$ for all $a, b \in \mathbb{N}$, then f is a constant function.

7: Find the largest $n \in \mathbb{N}$ such that whenever $a_1, \ldots, a_7 \in \{1, 2, \ldots, n\}$, there exist distinct $S, T \subseteq \{1, 2, \ldots, 7\}$ such that $\displaystyle\sum_{i \in S} a_i = \sum_{j \in T} a_j$.

8: A fair die is rolled repeatedly and independently. Let X be the number of times the die is rolled until the pattern 11 appears. Let Y be the number of times the die is rolled until the pattern 12 appears. Find the expected values of X and Y.

9: Let $k \in \mathbb{N}$ and let $L_k = \operatorname{lcm}(1, 2, \ldots, k)$. Let $n \in \mathbb{N}$ with $n \geq k + L_k$. Prove that $\displaystyle\binom{n}{k}$ is divisible by $\displaystyle\prod_{i=0}^{k-1} \frac{n - i}{\gcd(n - i, L_k)}$ and conclude that $\displaystyle\binom{n}{k}$ has at least k distinct prime divisors.

10: Suppose the set $\mathbb{Z}_{\geq 0}$ of non-negative integers is partitioned into finitely many arithmetic progressions of the form $a_i\mathbb{Z}_{\geq 0} + b_i$ with $i = 1, \ldots, n$, $b_i \geq 0$ and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. Prove that $a_n = a_{n-1}$.

11: Let $n \in \mathbb{N}$ and let $x_{ij} \in [0,1]$ for $i, j = 1, \ldots, n$. Prove that

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{n} x_{ij}\right) + \prod_{i=1}^{n}\left(1 - \prod_{j=1}^{n}(1 - x_{ij})\right) \geq 1.$$

12: Let a, b be coprime integers at least 2. Prove that there are infinitely many primes p such that the p-adic valuation $\nu_p(a^{p-1} - b^{p-1})$ is odd.

Week 3: Hints

1: Split into two cases: a is odd and a is even.

2: Let $c_n = a_n/b_n$. Express $\frac{a_n}{a_n+b_n}$ in terms of c_n.

3: Suppose for a contradiction that it is bounded. Let a and b be the minimum and maximum of $f^{-1}((0,0))$. What can $f((-\infty,a))$, $f([a,b])$ and $f((b,\infty))$ be?

4: Compute $(AB)^2$.

5: It shouldn't be hard to guess what A and B should be.

6: Suppose $f(a) < f(b)$, then $f(a) < f(ab+a+b) < f(b)$.

7:

8: Let Z be the number of times for the first 1. Then $E(X) = \frac{1}{6}(E(Z)+1) + \frac{5}{6}(E(Z)+1+E(X))$. Let W be the number of times for the first non-1. Relate Y to Z and W similarly.

9: Do some estimates with ν_p.

10: Generating functions!

11: Let x_{ij} be the probability that the (i,j)-entry of a random $n \times n$ matrix is 1 and let $1 - x_{ij}$ be the probability that it is 0.

12: Consider prime divisors of $x^{2^{k-1}} + y^{2^{k-1}}$.