Week 2: Assorted Problems

1: Let a, b be integers such that $\gcd(a, b) = 1$. Prove that $\gcd(a + b, ab) = 1$.

2: Find all primes p for which $(2^{p-1} - 1)/p$ is a square.

3: Suppose $x, y, z \geq 0$. Prove that $\dfrac{x^3 + y^3 + z^3}{3} \geq xyz + \dfrac{3}{4} \left|(x - y)(y - z)(x - z)\right|$.

4: Consider the sequence (a_n) of real numbers defined by $a_0 = -1$ and $\displaystyle\sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0$. Prove that $a_n > 0$ for all $n \geq 1$.

5: Let k be a fixed positive integer. Let $a_0 = 0$ and $a_{n+1} = ka_n + \sqrt{(k^2 - 1)a_n^2 + 1}$ for $n \geq 0$. Prove that $\dfrac{a_n}{2k} \in \mathbb{Z}$ for every $n \geq 0$.

6: Let $0 < r < 2$ be a real number. Define $I(t) = \displaystyle\int_0^\infty \frac{1 - \cos(tx)}{x^{r+1}} dx$.

(a) Prove that $I(t) = |t|^r I(1)$ for every $t \in \mathbb{R}$.

(b) Prove that for any real numbers $t_1, \ldots, t_n$, and any $r \in [0, 2]$, we have $\displaystyle\sum_{i,j} |t_i - t_j|^r \leq \sum_{i,j} |t_i + t_j|^r$.

7: An irrational number in $(0, 1)$ is *funny* if its first four decimal digits are the same. For example, $0.1111 + e/100000$ is funny. Prove that 0.1111 is not a sum of 1111 funny numbers and every number $x \in (0, 1)$ can be written as a sum of 1112 distinct funny numbers.

8: A subset of k elements of the set $\{1, 2, \ldots, 2022\}$ is selected randomly. Prove that the probabilities that the sum of the elements of the selected subset is congruent to 0 or 1 or 2 mod 3 are the same if and only if $k \equiv 1$ or $2 \bmod 3$.

9: Prove that there does not exist a rational number $\alpha \in (0, 1)$ such that $\cos(\pi\alpha) = \dfrac{-1 + \sqrt{17}}{4}$.

10: Let a be a fixed positive integer. Prove that the equation $n! = a^b - 1$ has only finitely many solutions in positive integers n, b.

11: Prove that $\displaystyle\lim_{n\to\infty} I_n$ exists where $I_n = \displaystyle\int_{[0,1]^n} \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \, dx_1 \cdots dx_n$.

12: Find all functions $f : \mathbb{N} \to \mathbb{Z}$ such that $f(a) \geq f(b)$ whenever $a \mid b$ and that for all $a, b \in \mathbb{N}$, $f(ab) + f(a^2 + b^2) = f(a) + f(b)$.

Week 2: Hints

1: What could a prime common divisor of $a+b$ and ab be?

2: Consider $p \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$ separately.

3: A bunch of AM-GM.

4: Proof by induction. Express a_{n+1} as a positive combination of $a_1, \ldots, a_n$.

5: Prove that $a_{n+2} - a_n = 2ka_{n+1}$.

6: For (b), $\cos(a-b) + \cos(a+b) = 2 \sin a \sin b$.

7: Find the largest $0.uuuu$ less than x, subtract and divide by 1112. Then adjust them to be irrational.

8: Let ω be the primitive cube root of unity. Consider the X^k-coefficient of $P(X) = (1 + X\omega)(1 + X\omega^2) \cdots (1 + X\omega^{2022})$.

9: Prove that the numbers of the form $\cos(2^n \pi \alpha)$ are all distinct. (Using the theory of cyclotomic extensions, one can show that the algebraic number $\cos(2\pi m/n)$ where $\gcd(m, n) = 1$ has degree $\phi(n)/2$.)

10: By the lifting exponent lemma $\nu_p(a^b - 1) \leq \nu_p(a^{p-1} - 1) + \nu_p(b)$.

11: Let $J_n = nI_n$. Prove that $J_{m+n} \leq J_m + J_n$.

12: Prove that $f(n)$ depends only on the prime divisors of n that are 3 mod 4.