Week 1: Assorted Problems

1: Suppose $\sin(1+\cos^2\alpha+\sin^4\alpha) = \dfrac{13}{14}$. Find $\cos(1+\sin^2\alpha+\cos^4\alpha)$.

2: Evaluate $\displaystyle\sum_{k=1}^{2023}\frac{k}{k^4+k^2+1}$.

3: Prove that there does not exist a non-constant rational function $f(x)$ (quotient of two polynomials) with coefficients in $\mathbb{R}$ such that $f\left(\dfrac{x^2}{x+1}\right)$ is a polynomial with coefficients in $\mathbb{R}$.

4: Prove that there exists an (countably) infinite square matrix such that every row and column contains every positive integer exactly once.

5: Let $r_1,\ldots,r_{2023}$ be the 2023 (complex) roots of $2024x^{2023}+2023x^{2022}+\cdots+2x+1$. Compute $\displaystyle\sum_{i=1}^{2023}\frac{1}{(1-r_i)^2}$.

6: Evaluate $\displaystyle\int_{-1}^{1}\frac{x^2}{1-2^x}\,dx$.

7: Suppose $f:[0,1]\to\mathbb{R}$ is a function that is continuous on $(0,1]$. Let $a,b>0$. Prove that $\displaystyle\lim_{x\to0^+}\frac{1}{x^a}\int_0^x t^{a-1}f(t)\,dt$ exists if and only if $\displaystyle\lim_{x\to0^+}\frac{1}{x^b}\int_0^x t^{b-1}f(t)\,dt$ exists.

8: Let $f(x)$ be a non-constant polynomial with integer coefficients. For any $n\in\mathbb{N}$, let a_n be the remainder when $f(3^n)$ is divided by n. Prove that the sequence $(a_n)_{n\geq1}$ is unbounded.

9: Let $n\geq 2$ be an integer. Prove that the number of n-tuples $(a_1,\ldots,a_n)$ of positive integers such that $(a_1!-1)\cdots(a_n!-1)-16$ is a perfect square, is $n(n-1)/2$.

10: Prove that for all integers a,b with $b\neq0$, there exists a positive integer n such that $\nu_2(n!)\equiv a \pmod b$, where $\nu_p(m)$ for a prime p and an integer m is the largest integer d such that $p^d\mid m$.

11: Let $(a_n)_{n\geq1}$ be a sequence of positive integers such that $\gcd(a_m,a_n)=a_{\gcd(m,n)}$ for any $m,n\in\mathbb{N}$. Prove that there exists a (unique) sequence of positive integers $(b_n)_{n\geq1}$ such that $a_n=\displaystyle\prod_{d\mid n}b_d$ for all $n\in\mathbb{N}$.

12: Let $f(x,y):\mathbb{R}^2\to\mathbb{R}$ be a differentiable function with continuous partial derivatives. Prove that there exist continuous functions $g_1(x,y),g_2(x,y):\mathbb{R}^2\to\mathbb{R}$ such that $f(x,y)=xg_1(x,y)+yg_2(x,y)$.

Week 1: Hints

1: $1 + \cos^2\alpha + \sin^4\alpha = 1 + \sin^2\alpha + \cos^4\alpha$. Be careful with the sign.

2: $k^4 + k^2 + 1 = (k^2 + k + 1)(k^2 - k + 1)$.

3: Write $f(x) = a(x)/b(x)$ and factor over $\mathbb{C}$.

4: Inductively construct a $2^n \times 2^n$ matrix such that every row and column contains $1, 2, \ldots, 2^n$ exactly once.

5: Find a polynomial that has $1/(1 - r_i)$ as roots.

6: Let $u = -x$.

7: Let $F(x) = \displaystyle\int_0^x t^{a-1} f(t)\,dt$. Then $t^{b-1}f(t) = t^{b-a}F'(t)$. Integration by part.

8: For any fixed $m \in \mathbb{N}$ and large enough prime p, $f(3^{pm}) \equiv f(3^m) \pmod{p}$.

9: Prove that $a_i \in \{2, 3\}$.

10: Legendre's formula gives $\nu_p(m) = (m - s_p(m))/(p - 1)$ where $s_p(m)$ is the sum of the digits of n when written in base p. So the problem reduces to finding a positive integer k and positive integers $x_1 < \cdots < x_k$ such that $2^{x_1} - 1 + \cdots + 2^{x_k} - 1 \equiv a \pmod{b}$.

11: Mobius inversion forces $b_n = \displaystyle\prod_{d|n} a_{n/d}^{\mu(d)}$ where μ is the Mobius function. Prove that $b_n \in \mathbb{Z}$ by showing that $b_n = a_n/\mathrm{lcm}(a_{n/p_1}, \ldots, a_{n/p_r})$ where $p_1, \ldots, p_r$ are the prime divisors of n.

12: FTC gives $f(x, y) - f(0, 0) = \displaystyle\int_0^x \frac{\partial f}{\partial x}(s, 0)\,ds + \int_0^y \frac{\partial f}{\partial y}(x, t)\,dt$. Then substitute $s = x\sigma$ and $t = y\tau$.