Week 7: Assorted Problems

1: Prove that for any integer $n > 1$, $n^5 + n^4 + 1$ is not prime.

2: Let a be a real number such that $5(\sin^3 a + \cos^3 a) + 3 \sin a \cos a = 0.04$. Prove that $5(\sin a + \cos a) + 2 \sin a \cos a = 0.04$.

3: Prove that $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+m+1} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{1}{k+n+1}$.

4: Find all positive integer solutions to $x! + y! + z! = 2^w$.

5: Prove that $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}^{-1} = 2$.

6: Let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $h(3x) + h(2x) + h(x) = 0$ for all $x \in \mathbb{R}$. Prove that $h = 0$.

7: Prove that for any integer $n \geq 3$, there exist odd integers x, y such that $2^n = 7x^2 + y^2$.

8: Given a set of positive real numbers such that the sum of the pairwise products is 1, prove that it is possible to remove one number so that the sum of the remaining numbers is less than $\sqrt{2}$.

9: Let A, B, C be $n \times n$ matrices with complex coefficients such that $ACB = xA + yB$ for some nonzero $x, y \in \mathbb{C}$. Prove that $ACB = BCA$.

10: Let $p(x)$ be a polynomial with integer coefficients such that $p(n) > n$ for every positive integer n. Define a sequence by $x_1 = 1$ and $x_{i+1} = p(x_i)$ for $i \geq 1$. Suppose for any positive integer m, there exists some r such that $m \mid x_r$. Prove that $p(x) = x + 1$.

11: Prove that the equation $\dfrac{x^7 - 1}{x - 1} = y^5 - 1$ has no integer solutions.

12: Prove the (in)famous 1988 IMO problem by induction on ab: Let a and b be positive integers, and suppose $q = \dfrac{a^2 + b^2}{ab + 1}$ is an integer. Then $q = \gcd(a, b)^2$.

Week 7: Hints

1: $n^2+n+1 \mid n^5+n^4+1$.

2: $x^3+y^3+z^3-3xyz = \frac{1}{2}(x+y+z)((x-y)^2+(y-z)^2+(z-x)^2)$.

3: $\dfrac{1}{k+m+1} = \displaystyle\int_0^1 x^{k+m}\,dx$.

4: Suppose $x \le y \le z$ and do a bunch of basic case analysis.

5: Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^{-1} = 2 + O\Big(\frac{1}{n}\Big)$.

6: Prove that for any $\epsilon > 0$, $\left|\dfrac{h(t)}{t}\right| < \epsilon$ for all $t \neq 0$.

7: Consider $7(x \pm y)^2 + (7x \mp y)^2$.

8: $a_1(S-a_1)+\cdots+a_n(S-a_n) \ge \min\{S-a_i\}\cdot S$, where $S = a_1+a_2+\ldots+a_n$.

9: Multiply by C on the right and factor.

10: Prove that $x_{n+1}-x_1 \mid x_i$ for some $i = 1, \ldots, n$.

11: Prove that any prime divisor of the LHS is 0 or 1 mod 7.

12: Suppose $a \le b$ and let $c = aq - b$.