Week 6: Assorted Problems

1: Let $a, b, c \in \mathbb{C}$ be such that $a + b + c = 0$ and $|a| = |b| = |c|$. Prove that $a^3 = b^3 = c^3$.

2: Prove that for $|x| > 1$, $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{1 + x^{2^n}} = \frac{1}{x-1}$.

3: Let p be an odd prime and let $a_1, \ldots, a_p$ be an arithmetic progression whose common difference is not divisible by p. Prove that there exists $i = 1, \ldots, p$ such that $a_1 a_2 \cdots a_p + a_i$ is divisible by p^2.

4: Let a, b be elements of a finite ring such that $ab^2 = b$. Prove that $bab = b$.

5: Find all continuous functions $f : [0, 1] \to \mathbb{R}$ such that $\displaystyle\int_0^1 f(x)\, dx = \frac{1}{3} + \int_0^1 f(x^2)^2\, dx$.

6: Compute $\displaystyle\int_0^{\pi} \frac{\sin nx}{\sin x}\, dx$ for any integer n.

7: Prove that $\displaystyle\lim_{n\to\infty} n^2 \int_0^{\frac{1}{n}} x^{x+1}\, dx = \frac{1}{2}$.

8: Let N be the number of integer solutions to the equation $x^3 - y^3 = z^4 - w^4$ with the property that $0 \le x, y, z, w \le 2022^{2022}$. Let M be the number of integer solutions to the equation $x^3 - y^3 = z^4 - w^4 + 1$ with the property that $0 \le x, y, z, w \le 2022^{2022}$. Prove that $N > M$.

9: A disk of radius R is covered by m rectangular stripes of width 2. Prove that $m \ge R$.

10: Prove that there exist integers a, b such that $b > a + 1$ and that for any integer $k = a + 1, a + 2, \ldots, b - 1$, either $\gcd(a, k) > 1$ or $\gcd(b, k) > 1$.

11: (Gelfand's Formula) For any $n \times n$ matrix A with complex coefficients, we define its spectral radius $\rho(A)$ to be the maximum of the absolute value of its eigenvalues. Prove that for any norm $||.||$ on $M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ (which are all equivalent), $\rho(A) = \displaystyle\lim_{k\to\infty} ||A^k||^{\frac{1}{k}}$.

12: For any positive integer n, let $f(n)$ be the number of permutations $a_1, \ldots, a_n$ of $1, \ldots, n$ such that $a_1 = 1$ and $|a_i - a_{i+1}| \le 2$ for $i = 2, \ldots, n-1$. Find $f(2022)$ mod 3.

Week 6: Hints

1: WLOG $a = 1, b = e^{i\beta}, c = e^{i\gamma}$.

2: $\displaystyle\sum_{n=0}^{m} \frac{2^n}{1+x^{2^n}} = \frac{1}{x-1} + \frac{2^{m+1}}{1-x^{2^{m+1}}}$.

3: $a_1, \ldots, a_p$ take all possible values mod p. Use Wilson's Theorem.

4: Prove that $b = b^k$ for some $k > 1$.

5: Consider $\displaystyle\int_0^1 (x - f(x^2))^2 \, dx$.

6: Let $I_n = \displaystyle\int_0^\pi \frac{\sin nx}{\sin x} \, dx$. Prove that $I_{n+2} = I_n$.

7: Prove that $\displaystyle\lim_{n\to\infty} n^2 \int_0^{\frac{1}{n}} (x^{x+1} - x) \, dx = 0$.

8: Let a_i be the number of integers (s, t) with $0 \le s, t \le 2022^{2022}$ such that $s^3 + t^4 = i$. Express N and M in terms of the a_i's.

9: Move to the sphere. Prove that the surface area of the part of the sphere of radius R bounded within two parallel planes of distance d is equal to $2\pi R d$.

10: $b - a - 1$ should have two distinct prime factors.

11: Prove (using Jordan blocks) that $\rho(A) < 1$ if and only if $\lim A^k = 0$.

12: Let $g(n)$ be the number of these permutations with $a_n = n$. Let $h(n) = f(n) - g(n)$. Find recursion formula about $g(n)$ and $h(n)$.