Week 5: Assorted Problems

1: Prove that there exist arithmetic progressions of arbitrary length consisting of pairwise coprime integers.

2: Let p be a prime and suppose $1 \leq k \leq p-1$. Find the number of k-element subset $\{a_1, \ldots, a_k\}$ of $\{1, \ldots, p\}$ such that $a_1 + a_2 + \cdots + a_k \equiv 0 \pmod{p}$.

3: Prove that 2002^{2002} can be written as a sum of 4 cubes but not a sum of 3 cubes.

4: Prove that every integer can be written in the form $\pm 1^2 \pm 2^2 \pm \cdots \pm m^2$ infinitely many ways.

5: Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) + f(y+z) + f(z+x) \geq 3f(x+2y+3z)$ for all $x, y, z \in \mathbb{R}$.

6: Let f be a continuous function such that $\lim_{x\to\infty} f(x)$ exists. Let a, b be positive real numbers. Evaluate the Frullani integral $\displaystyle\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx$.

7: Find the expected value of the square of the distance between two randomly chosen points on the unit sphere.

8: Suppose P_1, P_2, P_3, P_4 are 4 distinct points on the hyperbola $xy = 1$ with x-coordinates x_1, x_2, x_3, x_4. Suppose P_1, P_2, P_3, P_4 also lie on a circle. Prove that $x_1 x_2 x_3 x_4 = 1$.

9: Consider positive real numbers $x_1, x_2, \ldots, x_n$ with $x_1 x_2 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} \frac{1}{n-1+x_i} \leq 1.$$

10: Let p be a prime. Let A be a $p \times p$ matrix whose (i,j)-coordinate is $\binom{i+j-2}{i-1}$. Prove that $A^3 \equiv I \pmod{p}$.

11: Let a_n denote the exponent of 2 in the numerator of $\displaystyle\sum_{i=1}^{n} \frac{2^i}{i}$ when written in the lowest form. For example, $a_1 = 1$, $a_2 = 2$, $a_3 = 2$, $a_4 = 5$. Prove that $\lim_{n\to\infty} a_n = \infty$.

12: Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that for any fixed $y \in \mathbb{R}$, $f(x+y) - f(x)$ is a polynomial in x. Prove that f is a polynomial.

Week 5: Hints

1: Factorials should show up somewhere.

2: $(a_1 + c) + \cdots + (a_k + c) \equiv ck \pmod{p}$.

3: $2002 = 10^3 + 10^3 + 1^3 + 1^3$.

4: $(m+1)^2 - (m+2)^2 - (m+3)^2 + (m+4)^2 = 4$.

5: Try values of x, y, z to make as many terms $f(0)$ as possible.

6: $\dfrac{f(ax) - f(bx)}{x} = \displaystyle\int_b^a f'(xt)\,dt$.

7: The line joining x and y and the line joining x and $-y$ are perpendicular.

8: There exist real numbers A, B, C such that $x_i^2 + \dfrac{1}{x_i^2} + Ax_i + \dfrac{B}{x_i} + C = 0$ for $i = 1, 2, 3, 4$.

9: Prove $\dfrac{x_i}{n - 1 + x_i} \geq \dfrac{x_i^{1-1/n}}{x_1^{1-1/n} + \cdots + x_n^{1-1/n}}$.

10:

11: The series equals $-\log(1-x)$ when $x = 2$ in $\mathbb{Z}_2$.

12: Consider $f(x+y+z) - f(x+y) - f(x+z) - f(x)$ in two ways.