Week 4: Assorted Problems

1: Prove that the equation $8x^4 + 4y^4 + 2z^4 = w^4$ has no positive integer solutions.

2: Choose n numbers $x_1, \dots, x_n$ uniformly and independently from $[0, 1]$. Find the expected value of $\max_{1 \le i \le n} x_i - \min_{1 \le i \le n} x_i$.

3: Prove that a convex polygon in the plane with a prime number of sides, all angles equal, and all sides of rational length, must be regular (i.e., all sides also have equal length).

4: Prove that for any positive integer a, the last digit of $a^{a^{a^n}}$ is independent on the positive integer n.

5: Prove that for any integer $n \ge 10$, there is a perfect cube strictly between n and $3n$.

6: Let x, y be positive integers such that $2x^2+x = 3y^2+y$. Prove that $x-y$, $2x+2y+1$ and $3x+3y+1$ are all perfect squares.

7: Prove that $\displaystyle\int_0^1 \frac{1}{\ln x} + \frac{1}{1-x}\,dx = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N \right)$.

8: For any positive integer k, let p_k denote the k-th prime. Prove that for any positive integer m, $p_1^m + \cdots + p_n^m > n^{m+1}$.

9: Prove that there is a constant $C > 0$ such that for $\lambda > 1$, $\displaystyle\int_0^\infty e^{-\lambda(x^3+x^5)}\,dx = C\lambda^{-1/3} + O(\lambda^{-1})$.

10: Let n and k be positive integers with $n \ge 3$. Let $p(x) = x^n + x^{n-1} + \cdots + x - k$.

(a) Prove that $p(x)$ has no repeated (complex) roots.

(b) Prove that if $k > n$, then $p(x)$ has at least one root with negative real part and nonzero imaginary part.

11: Prove that there exist coprime integers L_0, L_1 such that the sequence defined by $L_n = L_{n-1} + L_{n-2}$ for $n \ge 2$ contains no prime numbers. (You may use the fact that the set of integers can be covered by finitely congruence classes with distinct moduli.)

12: Let $f(x)$ be a real-valued function with continuous third derivatives such that $f(x), f'(x), f''(x), f'''(x)$ are all positive for all x. Suppose $f'''(x) \le f(x)$ for all x. Prove that $f'(x) < 2f(x)$ for all x.

Week 4: Hints

1: $w = 2t$.

2: Find $\Pr(\min \le a, \max \le b)$ and take $\partial a \partial b$ for the joint probability density function.

3: $a_1 + a_2\zeta_p + \cdots + a_p\zeta_p^{p-1} = 0$.

4: $\phi(10) = 4$.

5: Show that if $\sqrt[3]{b} - \sqrt[3]{a} > 1$, then there is a perfect cube strictly between a and b.

6: Consider $(x-y)(2x+2y+1)$.

7: Set $x = e^{-t}$ and then expand. Handle $\dfrac{e^{-t}}{t}$ via $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-nt} - e^{-(n+1)t}}{t}$.

8: $\left(\dfrac{a_1^m + \cdots + a_n^m}{n}\right) \ge \left(\dfrac{a_1 + \cdots + a_n}{n}\right)^m$.

9: Consider $\displaystyle\int_0^\infty e^{-\lambda x^3} - e^{-\lambda(x^3+x^5)}\,dx$.

10: Consider $p(x)(x-1)$.

11: $L_n F_{c-1} + L_{n+1}F_c = L_{n+c}$. So if $p \mid F_c$ and $p \mid L_n$, then $p \mid L_{n+kc}$ for all $k \in \mathbb{Z}$.

12: Prove that $f'(0) + f''(0)x + f(0)x^2/2$ and $f(0) + f'(0)x + f''(0)x^2/2$ are always positive.