Week 3: Assorted Problems

1: Let $n \geq 2$ be an integer. Prove that $\displaystyle\prod_{k=0}^{n} \binom{n}{k} \leq \left(\frac{2^n - 2}{n-1}\right)^{n-1}$.

2: Let $P(x)$ be a polynomial with real coefficients such that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exist polynomials $Q(x)$ and $R(x)$ such that $P(x) = Q(x)^2 + R(x)^2$.

3: Let p be a prime. Let $f(x) = x^{p-1} + \cdots + 1$ Prove that $f(a) \equiv p \pmod{p^2}$ for any $a \equiv 1 \pmod{p}$.

4: An infinite checkerboard is colored black and white so that every 2×3 block contains exactly two white squares. How many white squares does a 2022×2021 block contain?

5: Prove that any collection of pairwise non-disjoint subsets of $\{1, \ldots, n\}$ has size at most 2^{n-1}.

6: Cagnus Marlsen trains by playing at least one game per day, but, to avoid exhaustion, no more than 13 games in any 7 consecutive days. Prove that there is a group of consecutive days in which Cagnus plays exactly 20 games.

7: Prove that for any positive integer m, there exists a positive integer N such that for all integer $r > N$, there exists positive integers $x_1, \ldots, x_r$ such that $\dfrac{1}{x_1^m} + \cdots + \dfrac{1}{x_r^m} = 1$.

8: Let A, B, C be three non-colinear points on $\mathbb{R}^2$ of distance at most d from each other. Prove that there are at most $O(d^2)$ points whose distances to A, B and C are all integers.

9: Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = \dfrac{f(x+n) - f(x)}{n}$ for all $x \in \mathbb{R}$ and all positive integers n.

10: Determine the number of permutations $a_1, a_2, \ldots, a_{2022}$ of $1, 2, \ldots, 2022$ such that $|a_i - i|$ is constant for $i = 1, \ldots, 2022$.

11: Prove that for any $x > 0$, $\displaystyle\int_0^\infty \frac{e^{-tx}}{1+t^2}\, dt = \int_0^\infty \frac{\sin t}{t+x}\, dt$.

12: For any positive integer k, find the smallest integer n such that there exist $n \times n$ matrices A, B such that $(AB)^k = 0$ and $(BA)^k \neq 0$.

Week 3: Hints

1: AMGM

2: The roots of $P(x)$ come in complex conjugate pairs.

3: $f(a+bp) \equiv f(a) + bpf'(a) \pmod{p^2}$.

4: Consider the distribution of white and black squares in a 3×3 block.

5: The collections $\{A_1, \ldots, A_m\}$ and $\{\bar{A_1}, \ldots, \bar{A_m}\}$ are disjoint.

6: There is a group of at most 20 days where Cagnus plays a multiple of 20 games.

7: Prove that if a solution exists for r, then it exists for $r + (a^m - 1)k$ for any positive integers m and k.

8: If $|d(P,A) - d(P,B)|$ is an integer less than d, then P lies on a union of $O(d)$ hyperbolae.

9: Prove $f''(x) = 0$.

10: Prove that $f(i) = a_i$ is an order 2 involution if it is not the identity.

11: Both sides satisfy $y'' + y = 1/x$.

12: Try some small values of k.