Week 2: Assorted Problems

1: Suppose $\alpha \in \mathbb{R}$ with $\cos\alpha \neq 1$. Evaluate $\dfrac{\sin(2022)+\sin(2022+\alpha)}{\cos(2022)-\cos(2022+\alpha)}$.

2: Let $a_1, \ldots, a_{2022}$ be real numbers in $(0,1)$. Define for $n \geq 2023$, $a_n = (a_{n-1}+a_{n-2}+\cdots+a_{n-2022})^{1/2023}$. Prove that $\lim_{n\to\infty} a_n$ exists.

3: Let $a, b, c, d > 0$. Find all possible values for

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}.$$

4: Given any increasing sequence $1 < a_1 < a_2 < \cdots < a_n$ of integers, prove that there exists $a_{n+1} > a_n$ such that $a_1 + \cdots + a_{n+1}$ divides $a_1^2 + \cdots + a_{n+1}^2$.

5: Prove that for any real numbers $a_1, \ldots, a_n$, we have $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_i a_j}{i+j-1} \geq 0$.

6: Consider the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_n = f_{n-2} + f_{n-1}$ for $n \geq 3$. Find all integers a, b such that $af_n + bf_{n+1}$ is another element of the Fibonacci sequence for all $n \geq 1$.

7: Let p be an odd prime. Compute $\displaystyle\sum_{k=0}^{\frac{p-1}{2}} \frac{(p-1)!}{(k!)^2(p-1-2k)!}$ modulo p.

8: Determine $\displaystyle\left\lfloor \prod_{n=2}^{2022} \frac{2n+2}{2n+1} \right\rfloor$ given that it is coprime to 2022.

9: Let $f(x) : [0,\infty) \to \mathbb{R}$ be a strictly decreasing function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\displaystyle\int_0^\infty \frac{f(x)-f(x+1)}{f(x)}\,dx$ diverges.

10: Let $M_1, \ldots, M_n$ be $m \times m$ real matrices such that $\{M_1, \ldots, M_n\}$ forms a group under matrix multiplication. Suppose $M_1 + \cdots + M_n$ has trace 0. Prove that $M_1 + \cdots + M_n = 0$.

11: For $x, y > 0$, define two sequences (x_n) and (y_n) by $x_1 = x$, $y_1 = y$ and $x_{n+1} = \frac{x_n+y_n}{2}$ and $y_{n+1} = \sqrt{x_n y_n}$. Prove that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \frac{\pi}{\displaystyle\int_0^\pi \frac{d\theta}{\sqrt{x^2\cos^2\theta + y^2\sin^2\theta}}}.$$

12: Find the smallest integer k such that for all quadratic polynomials $P(x)$ with integer coefficients, at least one of the integers $P(1), \ldots, P(k)$ has a 0 digit in base 2.

Week 2: Hints

1: What is the derivative of $\dfrac{\sin(x)+\sin(x+\alpha)}{\cos(x)-\cos(x+\alpha)}$?

2: Just do it.

3: Bound S between 1 and 2.

4: Given $1 < u < v$, set $x = v + u^2 - u$. Then $u + x$ divides $v + x^2$.

5: Integrate $\left(\sum_{i=1}^{n} a_i x^{i-1}\right)^2$.

6: $f_m f_n + f_{m+1} f_{n+1} = f_{m+n+1}$.

7: This is the constant term in $\left(x + \dfrac{1}{x} + 1\right)^{p-1}$.

8: Bound $\left(\dfrac{2n+2}{2n+1}\right)^2$ by something that telescopes.

9: Consider $\displaystyle\int_n^{n+m} \frac{f(x) - f(x+1)}{f(x)}\, dx$ for any $n, m \in \mathbb{N}$.

10: If T denotes $M_1 + M_2 + \ldots + M_n$, then $T^2 = nT$.

11: Let $f(x,y) = \displaystyle\int_0^{\pi} \frac{d\theta}{\sqrt{x^2\cos^2\theta + y^2\sin^2\theta}}$. Prove that $f(x,y) = f(\frac{x+y}{2}, \sqrt{xy})$. This is known as the Arithmetic-Geometric-Mean of x and y.

12: There aren't many ways $\pm(2^m - 1)$ could fit in an arithmetic sequence.