Week 1: Assorted Problems

1: Find the minimum of $\dfrac{\sin^3 x}{\cos x} + \dfrac{\cos^3 x}{\sin x}$ for $0 < x < \pi/2$.

2: Prove that for any positive integer n, the fraction

$$\frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)}$$

is of the form $a/2^b$ where a is an odd integer and $b < 2n$.

3: Let S be the set of 3×3 matrices with entries $0, 1, -1$. Prove that

$$\{\det(A) \colon A \in S\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}.$$

4: Define a sequence a_n by $\sum_{d|n} a_d = 2^n$. Prove that $n \mid a_n$ for all $n \geq 1$.

5: Let p be a prime and k be an integer. Find $\sum_{x \in \mathbb{F}_p \setminus \{0\}} x^k$, where $\mathbb{F}_p$ is the finite field of p elements.

6: Let S be a finite set of points on the plane. Suppose S has at least 2 points. Prove that there exists a point p on the plane whose maximum distance to the points of S is achieved by 2 different points in S. That is, there exists $u, v \in S$ and $p \in \mathbb{R}^2$ with $u \neq v$ and $d(p, u) = d(p, v) = \max_{x \in S} d(p, x)$.

7: Consider a sequence x_n with the property $x_{n+1} = 2x_n - n^2$ for $n \geq 0$. For which values of x_0 are all the terms of the sequence positive?

8: Find two primes less than 500 that divide $20^{22} + 1$.

9: Evaluate $\displaystyle\int_0^\infty \frac{t^2 - \sin^2 t}{t^4}\, dt$.

10: Let $f(x) : [0, \infty) \to \mathbb{R}$ be a continuous function such that

$$f(x)\left(f(x) - \frac{1}{x}\int_0^x f(t)\, dt\right) \geq (f(x) - 1)^2.$$

Prove that $f(x)$ is the constant function 1.

11: Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}$.

12: Find the number of functions $f(x)$ from non-negative integers to non-negative integers such that $f(f(n)) = n + 2022$ for every non-negative integer n.

Week 1: Hints

1: $\{\sin^3 x, \cos^3 x\}$ and $\{1/\cos x, 1/\sin x\}$ are oppositely sorted.

2: The given fraction equals $\frac{1}{2^{2n}}\binom{2n}{n}$.

3: How can 2×2 matrices have determinant 2.

4: a_d is the number of non-repeated binary words of length d. a_n/n is also the number of irreducible polynomials of degree n over $\mathbb{F}_2$.

5: Change the sum into a geometric sum involving a primitive root in $\mathbb{F}_p^\times$. Also, the sum is invariant under multiplying by a^k for any $a \in \mathbb{F}_p^\times$.

6: Take a circle enclosing S and start shrinking it.

7: If u_n and v_n are two sequences with the given property, what does $v_n - x_n$ look like?

8: Consider the order of 20 in $(\mathbb{Z}/p\mathbb{Z})^\times$.

9: $\displaystyle\int_0^\infty s^3 e^{-st}\, ds = \frac{3!}{t^4}$.

10: Consider $\displaystyle F(x) = \frac{1}{x}\int_0^x f(t)\, dt - 1$.

11: $\mathrm{Im}(-\log(1 - e^i))$.

12: f gives an order 2 involution on $\mathbb{Z}/2022\mathbb{Z}$.