Week 9: Assorted Problems

1: For any positive integer X, let $N(X)$ denote the number of integer solutions to $u^2 + uv + v^2 = X$. Prove that $6 \mid N(X)$ for all positive integers X.

2: Let $f : \mathbb{Q} \to \mathbb{R}$ be a function with $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{Q}$. Prove that f is constant.

3: Let u, v, w be three vectors in $\mathbb{R}^2$ of length 1. Prove that there is choice of $\pm$'s such that the length of $\pm u \pm v \pm w$ is at most 1.

4: Let $f(x, y)$ and $g(x, y)$ be continuous functions on the unit disk $D = \{(x, y)\colon x^2 + y^2 \leq 1\}$. Suppose $f > g$ on the boundary of D. Suppose f and g have continuous second partial derivatives on the interior of D with

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \leq e^f, \qquad \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} \geq e^g.$$

Prove that $f \geq g$ everywhere on D.

5: Prove that the positive integers that cannot be expressed as $\operatorname{lcm}(a, b) + \operatorname{lcm}(b, c) + \operatorname{lcm}(a, c)$ for some positive integers a, b, c are exactly the powers of 2.

6: (Pfaffian) Let $A = (a_{ij})$ be an $2n \times 2n$ skew symmetric matrix, that is $a_{ij} = -a_{ji}$ for all i, j. Prove that $\det(A)$, as a polynomial in a_{ij}'s is the square of a polynomial in the a_{ij}'s.

7: Let $g(x) \in \mathbb{C}[x]$ be a quadratic polynomial with complex coefficients such that $g(x) - x$ has two distinct roots. Prove that there does not exist a function $f : \mathbb{C} \to \mathbb{C}$ such that $g(x) = f(f(x))$ for all $x \in \mathbb{C}$.

8: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $|f(x)| \leq 1$ and $f'(x) > 0$ for all x. Suppose $0 < \alpha < \beta$. Prove that

$$\lim_{n\to\infty} \int_\alpha^\beta f'(nx - \frac{1}{x})\, dx = 0.$$

9: For any $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define for any positive integer p,

$$||v||_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \qquad \text{and} \qquad ||v||_\infty = \max_{1\leq i\leq n} |x_i|.$$

Prove that the series $\sum_{p=1}^{\infty}(||v||_p - ||v||_\infty)$ converges if and only if there is exactly one of the x_i's with the maximal absolute value.

10: Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $\displaystyle\int_0^1 f(x)\,dx = 1$. Prove that

$$\lim_{n\to\infty} \frac{n}{\ln n} \int_0^1 x^n f(x^n) \ln(1-x)\,dx = -1.$$

11: Find all integers r such that there are at least two pairs of positive integers (m,n) such that $2^m = n! + r$.

12: For any positive integer n, let

$$a_n = \sum_{k=1}^{n} \frac{k}{\gcd(n,k)}, \qquad b_n = \sum_{k=1}^{n} \frac{n}{\gcd(n,k)}.$$

(a) Prove that $2a_n = 1 + b_n$ for all $n \geq 1$.

(b) Prove that for any $s > 3$, $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \frac{\zeta(s)\zeta(s-2)}{\zeta(s-1)}$.