Week 8: Assorted Problems

1: Let $\mathbb{N}$ denote the set of positive integers. Prove that $f(x) = x$ is the only function from $\mathbb{N}$ to $\mathbb{N}$ for which $x^2 + f(y)$ divides $f(x)^2 + y$ for all $x, y \in \mathbb{N}$.

2: Let $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ and $b_n = \sum_{k=1}^{n} \frac{1}{(2k-1)^2}$. Prove that $\lim_{n\to\infty} n\left(\frac{b_n}{a_n} - \frac{3}{4}\right) = \frac{3}{\pi^2}$.

3: Let $p \equiv 3 \pmod 4$ be a prime. Let S be a set of $p-1$ consecutive positive integers. Prove that S cannot be partitioned into two subsets, each having the same products of the elements.

4: Prove Catalan's identity: Let $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ be the usual Fibonacci sequence. Then for any positive integers r, n with $r < n$,

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2.$$

5: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous with $xf(x) = 2\int_{x/2}^{x} f(t)\,dt + \frac{x^2}{4}$. Prove that $f(x) = x + C$ for some constant C.

6: Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be real numbers with $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 < \pi$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$. Prove that $\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3 + \cos\alpha_4 < \frac{1}{2}$. Note also that $\frac{1}{2}$ is the best possible upper bound.

7: Let A be an $n \times n$ matrix all of whose entries are ± 1. Prove that $|\det(A)| \leq \frac{2}{3}n!$.

8: Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = 1$. Prove that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 1.$$

9: Suppose $f(x, y)$ has continuous partial derivatives on the unit disk $D = \{(x, y) \colon x^2 + y^2 \leq 1\}$. Suppose $f(x, y) = 1$ on the boundary of D and that $\max_{(x,y)\in D} \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2 = 1$. Prove that

$$\left|\iint_D f(x, y)\,dx\,dx\right| \leq \frac{4}{3}\pi.$$

10: Let $ABCD$ be a convex quadrilateral with $AD = BC$. Let P be the intersection of the diagonals AC and BD. Let K and L be the circumcenters of the triangles PAD and PBC respectively (the circumcenter of a triangle is the center of the circle passing through the vertices of the triangle). Prove that the midpoints of AB, CD, and KL are colinear.

11: Prove that $\displaystyle\int_0^\infty \frac{\sin^2 x - x\sin x}{x^3} = \frac{1}{2} - \ln 2.$

12: Let $k > 1$ be any integer. Prove that the function $f(n) = n^k - n!$ is injective on $\mathbb{N}$.