Week 7: Assorted Problems

1: Prove the exponent of the prime number p in the prime factorization of $n!$ equals $\dfrac{n - s_p(n)}{p-1}$ where $s_p(n)$ is the sum of the digits of n when written in base p.

2: Suppose the series $\displaystyle\sum_{n=1}^{\infty} a_n x_n$ converges for all sequences (x_n) such that $\displaystyle\lim_{n\to\infty} x_n = 0$. Prove that the series $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges.

3: Consider two bags where the first bag has $m-1$ white balls and 1 black ball and the second bag has m white balls. Randomly pick one ball from each bag and place it into the other bag. Repeat this process n times. Compute the probability thst the black ball is in the first bag.

4: Suppose $f(x,y); [0,1]\times[0,1] \to \mathbb{R}$ have continuous second partial derivatives with $f(x,0) = f(0,y) = 0$ for all x, y. Prove that

$$\left| \int_0^1 \int_0^1 f(x,y)\, dxdy \right| \leq \frac{1}{4} \max_{0\leq x,y\leq 1} \left| \frac{\partial^2 f}{\partial x \partial y} \right|.$$

5: Prove that the equation $x^4 = y^2 + z^2 + 4$ has no integer solutions.

6: Let A be an $m \times n$ matrix, B be an $n \times p$ matrix and let C be a $p \times q$ matrix. Prove that

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank}(B) \leq \operatorname{rank}(ABC).$$

7: Let G be a group such that $(xy)^2 = (yx)^2$ for every $x, y \in G$. Prove that $(xyx^{-1}y^{-1})^2 = e$, where e is the identity of G.

8: Let $n > 1$ be an integer and let $f(x) = \displaystyle\int_0^x e^{-t} \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}\right) dt$. Prove that $f(x) = \dfrac{n}{2}$ has a solution in $(n/2, n)$.

9: Prove the Combinatorial Nullstellensatz: Let F be a field and let $f \in F[x_1, \ldots, x_n]$ be a polynomial. Let $S_1, \ldots, S_n$ be nonempty subsets of F such that $f(s_1, \ldots, s_n) = 0$ for all $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$. Let $g_i(x_i) = \prod_{s_i \in S}(x_i - s_i)$. Prove that there exist polynomials $h_1, \ldots, h_n \in F[x_1, \ldots, x_n]$ such that $f = g_1h_1 + \cdots + g_nh_n$ and $\deg(h_i) \leq \deg(f) - \deg(g_i)$ for all i.

As a consequence, prove that if $\deg(f) = t_1 + \cdots + t_n$ and $t_i < |S_i|$ for all i and the $x_1^{t_1} \cdots x_n^{t_n}$-coefficient of f is nonzero, then $f(s_1, \ldots, s_n) \neq 0$ for some $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$.

10: Find a nonzero polynomial $F(x, y, z)$ of the smallest degree such that $F(a, b, c) = 0$ for all integers $a, b, c \in [1, n]$.

11: Prove that the number of squarefree positive integers less than or equal to N is at least $\frac{6N}{\pi^2} - 2\sqrt{N} - 1$.

12: For any positive integer n, let a_n denote the product of the first n primes. Prove that every integer less than a_n can be written as a sum of at most $2n$ distinct divisors of a_n.