Week 6: Assorted Problems

1: Prove that the equation $x^2 + y^2 = z^n$ has positive integer solutions for all positive integers n.

2: Let $f(x) : [a.b] \to \mathbb{R}$ be a positive increasing function. Let $x_n \in [a, b]$ be such that $f(x_n)^n = \frac{1}{b-a}\int_a^b f(x)^n\,dx$ for $n \geq 1$. Prove that $\lim_{n\to\infty} x_n = 1$.

3: Compute the integral $\int_0^\infty \frac{u}{1+e^u}\,du$.

4: Consider two $n \times n$ matrices A, B with real coefficients and $AB = A + B$. Suppose some positive power of A is 0. Prove that $\det(B + 2021A) = \det(B)$.

5: Let a be an irrational number and let b be any real number. Prove that if $c, d \in \mathbb{R}$ such that $\lfloor an + b\rfloor = \lfloor cn + d\rfloor$ for any positive integer n, then $a = c$ and $b = d$.

6: Let G be a group with elements a, b, c, d such that $ab = ba = cd = dc$. Suppose the orders of a, b, c, d satisfy $o(a) = o(c) = n$ and $o(b) = o(d) = m$ with $\gcd(n, m) = 1$. Prove that $a = c$ and $b = d$.

7: Find the real cubic polynomial $p(x)$ such that $p(x) \leq x^4$ on $[0, 1]$ and such that $\int_0^1 |x^4 - p(x)|\,dx$ is minimal among all such cubic polynomials.

8: Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - y^2) = xf(x) - yf(y)$ for all $x, y \in \mathbb{R}$.

9: Let $A = (a_{ij})$ be an $n \times n$ invertible matrix with real entries and $n \geq 2$. Prove that there exist integers $x_1, \ldots, x_n$ not all 0 such that

$$\sum_{i=1}^{n} |a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n| \leq \sqrt[n]{n! \cdot \det A}.$$

10: Prove Hilbert's inequality: For any real numbers $a_1, \ldots, a_n$,

$$\sum_{1\leq i,j\leq n} \frac{a_i a_j}{i+j} \leq \pi \sum_{i=1}^{n} a_i^2.$$

11: Prove that if n is an integer at least 2, then $2^n - 1 \nmid 3^n - 1$.

12: For any positive integer n, let $f(n)$ denote the largest prime divisor of n. Let (a_n) be an increasing of positive integers. Prove that the set $\{f(a_i + a_j)\colon i \neq j\}$ is infinite.