Week 5: Assorted Problems

1: Let m, n be positive integers. Let p be the product of all the prime divisors of $\gcd(m, n)$. Prove that $\frac{\phi(mn)}{\phi(m)\phi(n)} = \frac{p}{\phi(p)}$ where $\phi(m)$ is the Euler-totient function.

2: Compute $\sum_{k=1}^{\infty} \frac{1}{\operatorname{lcm}(k, 10)^2}$.

3: Suppose $f(x) : [0, 1] \to \mathbb{R}$ is continuous. Prove that $\int_0^1 e^{f(x)}\,dx \int_0^1 e^{-f(y)}\,dy \geq 1$.

4: Find monic quadratic polynomials $P(x), Q(x), R(x)$ such that $P(Q(x)) = (x-1)(x-3)(x-5)(x-7)$ and $Q(R(x)) = (x-2)(x-4)(x-6)(x-8)$.

5: Prove Flett's Theorem: Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Show that if $f'(a) = f'(b)$, then there exists $c \in (a, b)$ such that $f(c) - f(a) = (c - a)f'(c)$.

6: Let p be an odd prime. Suppose $a_0, a_1, \ldots, a_{p+1}$ are integers such that $(x^2 - x + 1)^{(p+1)/2} = \sum_{n=0}^{p+1} a_n x^n$. Compute the remainder when $a_0^2 + a_1^2 + \cdots + a_{p+1}^2$ is divided by p.

7: Let $f(x) : [0, 1] \to \mathbb{R}$ be differentiable with $f(0) = f(1) = 0$. Prove that

$$\left(\int_0^1 x f(x)\,dx\right)^2 \leq \frac{1}{45} \int_0^1 (f'(x))^2\,dx.$$

8: Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix over $\mathbb{R}$.

(a) If $\operatorname{Tr}(A) = 0$, then there exists an invertible matrix P such that the diagonal entries of $P^{-1}AP$ are all 0.

(b) For any matrix $A \in M_n(\mathbb{R})$, prove that there exist a real number λ and two nilpotent matrices A_1 and A_2 such that $A = \lambda I_n + A_1 + A_2$. Here I_n denotes the $n \times n$ identity matrix and a matrix is nilpotent if some positive power of it is 0.

9: Let $c \in (0, 1)$ and $x_1 \in (0, 1)$ with $x_1 \neq c(1 - x_1^2)$. Define $x_n = c(1 - x_{n-1}^2)$ for $n \geq 2$. Prove that the sequence (x_n) converges if and only if $c \in (0, \sqrt{3}/2)$.

10: Let $f(x) = a_n x^n + \cdots + a_0$ be a degree n polynomial with coefficients in $\mathbb{R}$ with $n \geq 2$. Suppose

$a_k = 0$ for some $k = 1, \ldots, n-1$ and $a_i \neq 0$ for all $i \neq k$. Suppose $f(x)$ has n distinct real roots. Prove that $a_{k-1}a_{k+1} < 0$.

11: Find the smallest prime number p such that there exist positive integers a, b such that $a^2 + p^3 = b^4$.

12: Let $p \geq 5$ be a prime. Prove that p^2 divides $\sum_{k=1}^{p^2-1} \binom{2k}{k}$.