Week 4: Assorted Problems

1: Find the smallest prime p such that $p + p^{-1} \equiv 25 \pmod{143}$.

2: Compute $\displaystyle\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{kn^{2k}}$.

3: Let $p(x)$ be a polynomial with positive real coefficients. Prove that the function $\ln(p(e^x))$ is concave up on $(0, \infty)$.

4: Let $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two nonzero linear maps such that $f_1 \neq \lambda f_2$ for any $\lambda \in \mathbb{C}$. Prove that for any $v \in \mathbb{C}^n$, there exists $v_1, v_2 \in \mathbb{C}^n$ such that $v = v_1 + v_2$ and $f_1(v) = f_1(v_1)$ and $f_2(v) = f_2(v_2)$.

5: Suppose an $a \times b$ rectangle can be tiled by $1 \times m$ and $n \times 1$ rectangles. Prove that $n \mid a$ or $m \mid b$.

6: Let m be a positive integer. Prove that for any positive integers n, ℓ, there exists an $m \times m$ matrix A such that

$$A^n + A^\ell = I_m + \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 0 & \cdots & 0 & 0 \\ 3 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m-1 & m-2 & m-3 & \cdots & 1 & 0 \\ m & m-1 & m-2 & \cdots & 2 & 1 \end{pmatrix}.$$

7: Does $\sin(n^2) + \sin(n^3)$ converge as $n \to \infty$?

8: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\sup_{x,y} |f(x + y) - f(x) - f(y)| < \infty$. Prove that there exists a real number α such that $\sup_{x} |f(x) - \alpha x| < \infty$.

9: Prove that the equation $x^8 = n! + 1$ has finitely many solutions in positive integers.

10: Find all polynomials $f(x)$ with integer coefficients such that $f(p) \mid 2^p - 2$ for any prime p.

11: Prove that any integer which can be written as the sum of squares of three rational numbers can also be written as a sum of squares of three integers.

12: Prove Cohn's irreducibility criterion: Let $b \geq 2$ and let p be a prime. Suppose $p = (a_n a_{n-1} \cdots a_0)_b$ in base-b. That is, $p = a_n b^n + \cdots + a_0$ with $0 \leq a_i \leq b - 1$. Then the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is irreducible over $\mathbb{Q}$.