Week 3: Assorted Problems

1: Let k be a positive integer. Prove that there exist integers m, n such that $k < m < n$ and $\gcd(m(2^m - 1), n(2^n - 1)) = 1$.

2: Suppose $A \subset \mathbb{N}$ with $\limsup_{N \Rightarrow \infty} \frac{\#(A \cap [1, N])}{N} > 0$. Prove that $\sum_{n \in A} \frac{1}{n}$ is divergent.

3: Prove that every sequence in $\mathbb{R}$ has non-increasing or a non-decreasing subsequence.

4: Prove Beaty's Theorem: Let α, β be two positive irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Prove that the sets $\{\lfloor n\alpha \rfloor : n \geq 1\}$ and $\{\lfloor n\beta \rfloor : n \geq 1\}$ form a partition of the set of positive integers.

5: Let $\phi(n)$ denote the Euler-totient function of n. Prove that $\sum_{n=1}^{\infty} \frac{\phi(n) t^n}{1 - t^n} = \frac{t}{(1-t)^2}$.

6: Let $f : [0, \infty) \to [0, \infty)$ be a differentiable function. Suppose $f(0) = 0, f'(x) \leq \frac{1}{2}$ and $\int_0^\infty f(x)\, dx$ converges. Prove that for any $\alpha > 0$,

$$\int_0^\infty (f(x))^\alpha \, dx \leq \left(\int_0^\infty f(x)\, dx \right)^{\frac{\alpha+1}{2}}.$$

7: Compute the integral $\int_0^1 \int_0^1 \int_0^1 \frac{1}{(1 + x^2 + y^2 + z^2)^2}\, dxdydz$.

8: Let $F_1 = F_2 = 1, F_{n+1} = F_n + F_{n+1}$ be the Fibonacci sequence. Let f be a polynomial of degree 1009 such that $f(k) = F_k$ for $k \in \{1011, \ldots, 2020\}$. Show that $f(2021) = F_{2021} - 1$.

9: Let $A_1, \ldots, A_{2n}$ be diagonalizable $n \times n$ matrices over $\mathbb{C}$. Suppose $A_i A_j = 0$ whenever $i < j$. Prove that at least n of $A_1, \ldots, A_{2n}$ are 0.

10: Let $B_1, \ldots, B_n$ be n boxes such that B_k contains 1 red ball and $k - 1$ white balls for every $k = 1, \ldots, n$. Take one ball from each box at random and let S_n denote the number of red balls. Prove that for any $\epsilon > 0$, $\lim_{n \to \infty} \text{Prob}(|S_n - 1| \geq \epsilon) = 0$.

11: Let (b_n) be a decreasing sequence of positive real numbers with $\lim_{n \to \infty} b_n = 0$. Let (a_n) be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n b_n$ converges. Prove that $\lim_{n \to \infty} (a_1 + a_2 + \cdots + a_n) b_n = 0$.

12: Prove the following statements:

(a) For any positive integer n, $(n-1)^2 \mid n^{n-1} - 1$.

(b) The only polynomial $f(n)$ with integer coefficients such that $f(n) \mid n^{n-1} - 1$ for all sufficiently large n is 1, $n-1$ or $(n-1)^2$.