Week 2: Assorted Problems

1: Compute $\lim_{n\to\infty} \sin^2(\pi\sqrt{n^{2022}+n^{1011}+1})$

2: Prove the interpolation formula: Let $f(x)$ be a polynomial of degree n. Then

$$f(n+1) = \sum_{k=0}^{n} f(k)\binom{n+1}{k}(-1)^{n-k}.$$

3: Prove that among any seven real numbers $x_1, \ldots, x_7$, there are two such that $0 \le \dfrac{x_i - x_j}{1 + x_i x_j} \le \dfrac{1}{\sqrt{3}}$.

4: Let n be a positive integer. Prove that the number of permutations σ on the set $\{1, \ldots, n\}$ such that $\sigma(i) \neq i$ for all $i = 1, \ldots, n$ is $n!\left(1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \cdots + (-1)^n \dfrac{1}{n!}\right)$.

5: Let α, β, A be real numbers. Prove that $\lim_{n\to\infty} \sin(n\alpha + \beta) = A$ if and only if $\sin\alpha = 0, \sin\beta = A$.

6: Suppose the set of non-negative integers is partitioned into a finite number of arithmetic progressions with starting terms $a_1, \ldots, a_n$ and common differences $r_1, \ldots, r_n$. Prove that

(a) $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \cdots + \dfrac{1}{r_n} = 1.$

(b) $\dfrac{a_1}{r_1} + \dfrac{a_2}{r_2} + \cdots + \dfrac{a_n}{r_n} = \dfrac{n-1}{2}.$

7: For $n \ge 0$, prove that $\displaystyle\prod_{k=0}^{n} \frac{(x+2+4k)^4+4}{(x+4k)^4+4} = \frac{(x+4n+3)^2+1}{(x-1)^2+1}$.

8: Let a and b be integers greater than 1. Prove that there is a multiple of a which contains all digits $0, 1, \ldots, b-1$ when written in base b.

9: Prove that $\lfloor (5\sqrt{5}+11)^{2n+1} \rfloor$ is even.

10: Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

11: Suppose $f : [-2, 2] \to \mathbb{R}$ is an even continuous function. Prove that

$$\int_1^2 f(x^3 - 3x)\,dx = \int_0^1 f(x^3 - 3x)\,dx.$$

12: Let m, n be positive integers and let k be a nonzero real number. Prove that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = k(f(x))^{2n} + x^{2m-1}$ for all $x \in \mathbb{R}$.