Week 10: Assorted Problems

**1:** Eight consecutive positive integers are partitioned into two subsets such that the sum of the squares in each subset is the same. Prove that the sum of elements in each subset is also the same.

**2:** Let $p$ be a prime and let $f(x_1, \ldots, x_{p-1})$ be a symmetric polynomial in $p-1$ variables. Suppose $f$ is homogeneous of degree $d$ with $p-1 \nmid d$. Prove that $p$ divides $f(1, 2, \ldots, p-1)$.

**3:** For any positive integer $m$, let $\nu_5(m)$ denote the highest power of 5 dividing $m$. Consider the usual Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that $\nu_5(F_n) = \nu_5(n)$ for all $n \geq 1$.

**4:** Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = 1 + x^2 + x^4 - x^3 - x^5$?

**5:** Prove that if $n$ is a positive integer not equal to 1 or 3, then for any permutation $\sigma$ of $\{1, \ldots, n\}$, we have

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} \notin \mathbb{Q}.$$

**6:** Let $n \geq 2$ be an integer and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ with

$$\sum_{i=1}^{n} a_i^2 = 1, \quad \sum_{i=1}^{n} b_i^2 = 1, \quad \sum_{i=1}^{n} a_i b_i = 0.$$

Prove that $\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leq n.$

**7:** Use the change of variable $x = u - v, y = u + v$ to compute the integral

$$\lim_{\epsilon \to 0} \int_0^{1-\epsilon} \int_0^{1-\epsilon} \frac{1}{1 - xy}\, dx dy$$

and use this to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

**8:** For any positive integer $n$, let $f(n)$ denote the largest prime divisor of $n$. Prove that $f(n^2+n+1) < f(n^2 - n + 1)$ and $f(n^2 + n + 1) > f(n^2 - n + 1)$ both happen infinitely often.

**9:** For any $\alpha \in (0, \pi/2)$, let

$$I(\alpha) = \int_0^{\infty} \frac{\ln(t^2 + 2(\sin \alpha)t + 1}{t^2 + 1}\, dt.$$

Prove that $I(\alpha) + I(-\alpha) = \pi \ln(2(1 + \cos\alpha))$.

**10:** Prove that $n$ distinct points, not all colinear, determine at least $n$ distinct lines.

**11:** (Continuation of Q1) Prove that the set $\{1, \ldots, n\}$ can be partitioned into two subsets such that the sum of elements in each subset is the same, the sum of squares in each subset is the same, and the sum of cubes in each subset is the same, if and only if $8 \mid n$ and $n \geq 16$.

**12:** Prove that $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^k = \frac{e}{e-1}$.