Week 1: Assorted Problems

1: Prove that for all $n \geq 3$, $n^{n+1} > (n+1)^n$.

2: Show that there exists a polynomial $f(x,y)$ in two variables whose image is the open interval $(0,\infty)$.

3: Evaluate $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} \frac{n}{n^2+j^2}$.

4: Let $f(x)$ be a continuous real-valued function on $[0,1]$ for which $\displaystyle\int_0^1 f(x)\,dx = 0$ and $\displaystyle\int_0^1 xf(x)\,dx = 1$. Prove that $|f(x)| > 4$ for some $x \in (0,1)$.

5: Prove that there are no four consecutive non-zero binomial coefficients $\binom{n}{r}, \binom{n}{r+1}, \binom{n}{r+2}, \binom{n}{r+3}$ in arithmetic progression.

6: Find the constant C such that $\displaystyle\int_0^x t^{2021} e^t \sin t\,dt = p(x)e^x \sin x + q(x)e^x \cos x + C$ for some polynomials $p(x)$ and $q(x)$.

7: Let $\{a_n\}$ and $\{b_n\}$ be positive real sequences such that $\displaystyle\lim_{n\to\infty} \frac{a_n}{n} = u > 0$ and $\displaystyle\lim_{n\to\infty} \left(\frac{b_n}{a_n}\right)^n = v > 0$. Prove that $\displaystyle\lim_{n\to\infty} b_n - a_n = u \log v$.

8: Prove Lucas's Theorem: Let m and n be positive integers and let p be a prime. Suppose $m = (m_k m_{k-1} \cdots m_1 m_0)_p$ and $n = (n_k n_{k-1} \cdots n_1 n_0)_p$ in base-p. Prove that $\displaystyle\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p}$.

9: Prove that $\displaystyle\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = 2^{1-n} n$.

10: Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for all positive integers n.

11: Consider the Chebyshev polynomials defined by $T_0(x) = 1$, $T_1(x) = x$ and $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$. Prove that for all $n \geq 2$,

$$\frac{1}{3} < \int_1^\infty T_n(x)^{-2/n}\, dx < \frac{1}{3}\sqrt[n]{4}.$$

12: Let p be a prime and let $n \geq 1$ be an integer. For any polynomial $f(x) \in \mathbb{F}_p[x]$, let $V(f) = \#\{f(a)\colon a \in \mathbb{F}_p\}$. Let $a_{n-1}, \ldots, a_2 \in \mathbb{F}_p$ be arbitrary. Prove that

$$\sum_{a_1 \in \mathbb{F}_p} V(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x) \geq \frac{p^3}{2p-1}.$$