

*The Faculty of Mathematics at the University of Waterloo
in association with
The Centre for Education in Mathematics and Computing
presents*

The Twenty-Second Annual Small c Competition

for First and Second Year Students

Monday 25 September 2023

Time: 1 hour

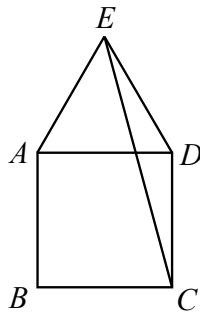
Calculators are permitted.

Instructions:

1. Do not open this booklet until you are told to do so.
10. You may use slide rules, abaci, rulers, compasses and paper for rough work. You may also use log tables; log cabins are not permitted. Protractors are also permitted, though contractors are not.
11. By Faculty policy, only fourth-year students are allowed to use scissors. (Of course, they can't run with them.) Thus, there are no scissors allowed on the Small c.
100. Any contestant carrying an Elongated Pentagonal Orthocupolarotunda must register it with a proctor.
101. You must **print your name and ID number on the response form**. No other information is needed but answers to the contest questions are highly recommended.
110. This is a multiple choice test. Each question is followed by five possible answers marked **A**, **B**, **C**, **D**, and **E**. Only one of these is correct. When you have decided on your choice, fill in the appropriate bubble on the response form.
111. Scoring: Each correct answer is worth 5 in Part A, 6 in Part B, and 8 in Part C.
There is *no penalty for an incorrect answer*.
Each unanswered question is worth 2, to a maximum of 20.
1000. Diagrams are *not necessarily drawn to scale*. They are intended as aids only.
1001. Als u dit kunt lezen, spreekt u het Nederlands.
1011. When a proctor instructs you to begin, you will have 111100 minutes of working time.
1011. Anyone overheard making a joke about the Toronto Maple Leafs will be immediately removed from the premises.
1100. The only website you may use during the contest is theonion.com.
1011. Data was scrambled during construction in the MC building. Try and find the flipped bit above.
1110. Turn off and put away your cell phones, tablets, laptops, desktops, satellites and quantum computers.
1111. Hint: The answer to at least one question is **B**.
10000. One bonus mark will be awarded to any contestant that gets Question 1 wrong and Question 25 right.
10001. Praising the Small c Competition on the subreddit [reddit.com/r/uwaterloo](https://www.reddit.com/r/uwaterloo) is permitted as of 7 p.m. tonight.
10010. Use of AI on this contest, while not expressly forbidden, will likely lead to hilariously inaccurate results.
10011. Due to an unfortunate virus, the 2020 edition of the Small c contest was deleted. If you recover this contest, please contact the CEMC immediately.
10100. Don't forget to turn off the Waterloo tap before you leave the Kitchener.

Part A

- A random student sees you wearing a Waterloo Math hoodie and asks you to calculate $\frac{1}{2} + \frac{2}{3} + \frac{3}{4}$, just to make sure you deserve to wear it. What is the answer to this unusual request?
(A) $\frac{4}{5}$ (B) $\frac{23}{12}$ (C) $\frac{2}{3}$ (D) $\frac{1}{2}$ (E) $\frac{1}{4}$
- The median of the five numbers $0.33, \frac{4}{5}, -2, -\frac{7}{6}, \sqrt{2}$ is
(A) 0.33 (B) $\frac{4}{5}$ (C) -2 (D) $-\frac{7}{6}$ (E) $\sqrt{2}$
- To prevent server overloads, the university has decided to create a virtual line for course registration. Sulaiman is waiting to register for courses. At 9:00 a.m. they are listed as having 12,400 people ahead of them. If 200 people are processed per minute, when will Sulaiman reach the front of the line?
(A) 10:22 a.m. (B) 9:52 a.m. (C) 10:12 a.m. (D) 9:42 a.m. (E) 10:02 a.m.
- In the interest of budgeting your time, you realize that this 60-minute contest has 25 questions: 10 questions in Part A, 10 questions in Part B, and 5 questions in Part C. On average, you are confident that you can answer each Part A question in 1 minute, and each Part B question in 2 minutes. How much time in minutes, on average, do you have to answer each Part C question?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7
- In the diagram shown, $ABCD$ is a square and AED is an equilateral triangle. What is the measure of $\angle ECD$?



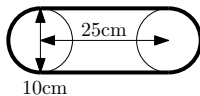
- (A) 15° (B) 30° (C) 25° (D) 5° (E) 10°
- Consider the following unnecessarily complicated system of equations:
$$5.67 + A = 5.7$$
$$5.7 + B = 6$$
$$6 + C = 20$$

What is the value of $A + B + C$?
(A) 4.33 (B) 4.6 (C) 14.33 (D) 14.6 (E) 14.96
 - For those times you run out of toiletries, Math C&D has started selling bottles of shampoo. Of the five sizes given here, which gives you the lowest cost per ounce?
(A) A 20-ounce bottle for \$1.80
(B) A 32-ounce bottle for \$3
(C) A 48-ounce bottle for \$5
(D) A 30-ounce bottle for \$2.50
(E) A 28-ounce bottle for \$2.50

8. After carefully crunching the numbers, Math Admissions determines that the number of applicants to the Math Faculty increases by 5% each year. How many years will it take for the number of applicants to be at least twice as much as the current year?

(A) 13 (B) 15 (C) 16 (D) 17 (E) 20

9. A conveyor belt fits tightly around two circular pulleys whose centres are 25 cm apart, as shown. Each pulley is 10 cm in diameter. How long is the conveyor belt, to the nearest cm?



(A) 93 cm (B) 113 cm (C) 45 cm (D) 61 cm (E) 81 cm

10. The *inflation rate* measures the rate of increase in an index function $p(t)$, indicating the price of certain goods at time t . A news headline reports: GROWTH IN INFLATION RATE SLOWING DOWN. Which function must be negative at the moment, according to this headline?

(A) $p(t)$ (B) $p'(t)$ (C) $p''(t)$ (D) $p'''(t)$ (E) $p''''(t)$

Part B

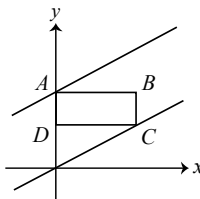
11. Suppose \circ is an operation on pairs of integers such that $(a, b) \circ (c, d) = (ac - a, bd - c)$. If $(x, y) \circ (2, 4) = (x, y) \circ (3, 2)$, then what is the value of $x - 2y$?

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

12. Kyle writes out a list of 30 consecutive even numbers, starting at 14, in order from least to greatest. He then removes every third number from the list, starting with the third. What is the average of the remaining numbers?

(A) 42 (B) 1290 (C) 450 (D) 84 (E) 30

13. Rectangle $ABCD$ has vertices A and D on the y -axis, with vertex A on the line $x - 2y + 8 = 0$ and vertex C on a parallel line passing through the origin, as shown. Given $CD = 3$, what is the area of rectangle $ABCD$?



(A) $\frac{9}{2}$ (B) 10 (C) $\frac{15}{2}$ (D) 15 (E) $\frac{45}{2}$

14. The first term in a *Gosling sequence* can be any integer from 1 to 10 inclusive. Beginning with the second term, each new term is generated using the previous term:

- If a term is even, divide it by 2 to get the next term.
- If a term is odd, triple it and add 1 to get the next term.

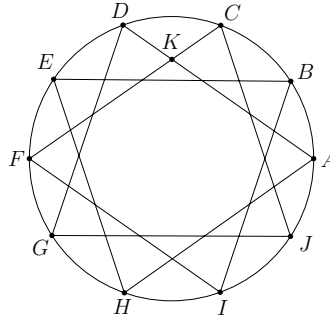
For example, the sequence 5, 16, 8, 4... shows the first four terms of a Gosling sequence that begins with 5. What is the greatest number that is part of a Gosling sequence?

(A) 16 (B) 34 (C) 52 (D) 100 (E) 232

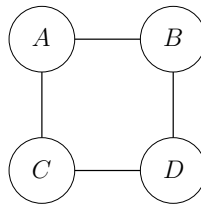
15. The lines $y = mx - b$ and $by = mx - 1$, where m and b are real numbers with $b \neq 1$ and $m \neq 0, -1$, intersect at a point. Determine the sum of the x and y coordinates of this intersection point, in terms of m and b .

(A) $\frac{1+b-m}{m}$ (B) $\frac{1+b+m}{m}$ (C) $\frac{-1+b-m}{m}$ (D) $\frac{1-mb+m}{m}$ (E) $\frac{1+mb+m}{m}$

16. A right triangle has sides of integer lengths $x + 2$, $3x - 2$, $3x - 4$. How many values of x make this statement true?
 (A) 0 (B) 1 (C) 2 (D) 3 (E) 4
17. The average (mean) of five positive integers is 45. The median value is 40. The unique mode is also 40. The range is 60. What is the greatest possible value any of these five integers can take?
 (A) 83 (B) 87 (C) 88 (D) 89 (E) 90
18. In the diagram, the points $ABCDEFGHIJ$ form a regular decagon. What is the measure of $\angle AKC$?



- (A) 56° (B) 60° (C) 72° (D) 76° (E) 80°
19. Consider a sequence of functions $f_k(x)$, where $f_1(x) = e^x$, and for $k \geq 2$, $f_k(x) = e^{f_{k-1}(x)}$. Which of the following correctly gives an expression for $\frac{d}{dx}(f_5(x))$?
 (A) $f_{15}(x)$
 (B) $f_1(x)$
 (C) $f_5(x) + f_4(x) + f_3(x) + f_2(x) + f_1(x)$
 (D) $f_5(x)$
 (E) $f_5(x)f_4(x)f_3(x)f_2(x)f_1(x)$
20. Four numbers from the set $\{1, 2, 3, 4, 5, 6, 7\}$ are placed in the four positions A, B, C and D shown. This is done in such a way that all of the integers from 1 to 13 can be constructed, one way, by adding between one and four adjacent numbers in the diagram (e.g. $A, A + B, A + C, A + B + D$, but not $A + D$), with any single number used at most once. Which number from the set cannot be placed in the diagram, in any solution to this problem?



- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Part C

21. The *Small c sequence* is a recursive sequence $\{u_n\}$ defined by $u_1 = 1$, $u_2 = 2$, and for $n \geq 2$,

$$u_{n+1} = \begin{cases} u_n + u_{n-1}, & \text{if } u_n + u_{n-1} \text{ is odd} \\ u_n - u_{n-1}, & \text{if } u_n + u_{n-1} \text{ is even} \end{cases}$$

What is the smallest index k for which $u_k > 100$?

- (A) 11 (B) 50 (C) 76 (D) 101 (E) No such index k exists.

22. A bag contains three marbles: one green, one yellow, and one blue. Anik draws from the bag six times with replacement. What is the probability that Anik draws the same marble colour exactly five times?

- (A) $\frac{4}{81}$ (B) $\frac{5}{729}$ (C) $\frac{5}{6}$ (D) $\frac{1}{4}$ (E) $\frac{2}{243}$

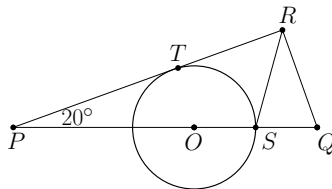
23. How many triples (a, b, c) are solutions to the equation $1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ given that $a, b,$ and c are positive integers, not necessarily distinct?

- (A) 6 (B) 10 (C) 14 (D) 18 (E) 22

24. If we take the 50 smallest solutions to $\sin(\ln(x^2)) = \sin(\ln(x))$ with $x \geq 1$ and calculate their sum, using radian measure for the angles, what is the result?

- (A) $\frac{e^{152\pi/3} + e^{26\pi} + e^{25\pi} - e^\pi - e^{2\pi/3} - 1}{e^{2\pi} - 1}$
 (B) $\frac{e^{109\pi/3} + e^{36\pi} + e^{107\pi/3} - e^{5\pi/3} - e^{\pi/3} - 1}{e^{2\pi} - 1}$
 (C) $\frac{e^{50\pi} - 1}{e^\pi - 1}$
 (D) $\frac{e^{104\pi/3} + e^{34\pi} + e^{33\pi} - e^\pi - e^{2\pi/3} - 1}{e^{2\pi} - 1}$
 (E) $\frac{e^{79\pi/3} + e^{77\pi/3} + e^{26\pi} + e^{25\pi} - e^{5\pi/3} - e^\pi - e^{\pi/3} - 1}{e^{2\pi} - 1}$

25. In $\triangle PQR$, $\angle RPQ = 20^\circ$, $\angle PRQ = 90^\circ$, O is a point on PQ and S is the midpoint of OQ . The circle with centre O and radius OS touches PR at T . The measure of $\angle QRS$ is



- (A) 20° (B) 30° (C) 35° (D) 45° (E) 55°