The Faculty of Mathematics at the University of Waterloo
in association with
The Centre for Education in Mathematics and Computing
presents

The Twentieth Annual
Small c Competition

for First and Second Year Students

Monday 27 September 2021

Time: 1 hour
Calculators are permitted.
Instructions:

1. Do not open this booklet until you are told to do so.

10. You may use slide rules, abaci, rulers, compasses and paper for rough work. You may also use log tables; log cabins are not permitted. Protractors are also permitted, though contractors are not.

11. By Faculty policy, only fourth-year students are allowed to use scissors. (Of course, they can't run with them.) Thus, there are no scissors allowed on the Small c.

100. Any contestant carrying an Elongated Pentagonal Orthocupolarotunda must register it with a proctor.

101. You must **print your name and ID number on the response form**. No other information is needed but answers to the contest questions are highly recommended.

110. This is a multiple choice test. Each question is followed by five possible answers marked **A**, **B**, **C**, **D**, and **E**. Only one of these is correct. When you have decided on your choice, fill in the appropriate bubble on the response form.

111. In the past, your response form was read only by a *dumb human*, who had undergone rigorous training in order to be able to recognize the letters **A** through **E**. Due to labour unrest, the dumb humans have been replaced by even dumber machines.

1000. Scoring: Each correct answer is worth 5 in Part A, 6 in Part B, and 8 in Part C.
There is *no penalty for an incorrect answer.*
Each unanswered question is worth 2, to a maximum of 20.

1001. Diagrams are *not necessarily drawn to scale.* They are intended as aids only.

1010. Als u dit kunt lezen, spreekt u het Nederlands.

1011. When a proctor instructs you to begin, you will have 111100 minutes of working time.

1100. Anyone overheard making a joke about the Toronto Maple Leafs will be immediately removed from the premises.

1001. The only website you may use during the contest is `theonion.com`.

1110. Data was scrambled during construction in the MC building. Try and find the flipped bit above.

1111. Turn off and put away your cell phones, tablets, laptops, desktops, satellites and quantum computers.

10000. Hint: The answer to at least one question is **B**.

10001. If you finish early, please Brexit the room quietly.

10010. Don't give up. As shown by the 2016 U.S. presidential election, anyone can win this thing.

10011. One bonus mark will be awarded to any contestant that gets Question 1 wrong and Question 25 right.

10100. Praising the Small c Competition on the subreddit `reddit.com/r/uwaterloo` is permitted as of 7 p.m. tonight.

10101. Due to an unfortunate virus, the 2020 edition of the Small c contest was deleted. If you recover this contest, please contact the CEMC immediately.

Part A

1. If $x = 4$ and $y = -3$, what is the value of $\dfrac{x^2 - xy}{xy - y^2}$?

   (A) $-\dfrac{4}{3}$ (B) $-\dfrac{3}{4}$ (C) $-\dfrac{1}{4}$ (D) $-\dfrac{1}{3}$ (E) $-\dfrac{1}{7}$

2. A rectangular hole has dimensions 3 cm by 4 cm. If you wish to place a round peg in this hole, what is the maximum possible radius of such a peg?

   (A) 1 cm (B) $\dfrac{3}{2}$ cm (C) 2 cm (D) $\dfrac{5}{2}$ cm (E) 3 cm

3. What is the mean of the median and the mode of the following set of numbers?

$$\{1, 2, 2, 2, 3, 3, 4, 4, 6\}$$

   (A) $\dfrac{3}{2}$ (B) 3 (C) $\dfrac{5}{2}$ (D) 2 (E) 7

4. The towns of Oolretaw and Otnorot are 100 km apart, and connected by a train line. At 6:00 a.m. one morning, a train sets off from Oolretaw moving towards Otnorot at 40 km/hr. At the same time, another train sets off from Otnorot towards Oolretaw at 60 km/hr. At what time, and where, do the two trains meet?

   (A) 6:30 a.m., 50 km from Oolretaw
   (B) 7:00 a.m., 40 km from Oolretaw
   (C) 7:40 a.m., 40 km from Oolretaw
   (D) 7:00 a.m., 60 km from Oolretaw
   (E) 8:30 a.m., 60 km from Oolretaw

5. Which of the following has the largest perimeter?

   (A) An isosceles right-angled triangle with hypotenuse 1 cm
   (B) An isosceles right-angled triangle with base and height 1 cm
   (C) An equilateral triangle with side length 1 cm
   (D) A square with side length 1 cm
   (E) A square with diagonal length 1 cm

6. On an analog clock, what is the acute angle between the minute hand and hour hand at 4:30?

   (A) $45°$ (B) $60°$ (C) $65°$ (D) $70°$ (E) $75°$

7. For all real numbers a and b with $a \neq 0$, $b \neq 0$, $a \neq b$, the quantity

$$\frac{a^{-2} - b^{-2}}{a^{-1} - b^{-1}}$$

   is equal to

   (A) $\dfrac{a + b}{ab}$ (B) $\dfrac{ab}{b - a}$ (C) $\dfrac{1}{a} - \dfrac{1}{b}$ (D) $\dfrac{1}{a + b}$ (E) None of these

8. At "Graphs R Us", a membership costs \$39 per year. This membership gives a customer 12% off all purchases that year. In the years 2016 through 2020, Asmita bought a membership and made purchases of \$39, \$112, \$320, \$360, and \$390, respectively, before any discounts were applied. In how many of these years was her total discount greater than the cost of the membership?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

9. In the currency of Planet Yafed, we have the following conversions:

   - 3 timpins equals 8 baznos,

   - 2 baznos equals 7 flutars, and

   - 4 flutars equals 27 picolets.

   Nef is visiting Planet Yafed and wants to buy an abacus that costs 31.5 picolets. Nef pays entirely in timpins. How many timpins does Nef pay?

   (A) 0.25 **(B)** 0.5 **(C)** 3 **(D)** 4.5 **(E)** 6

10. Xela has ten balls: 4 white, 3 grey, 2 blue, and 1 red. All balls of the same colour are identical. Xela selects two of these balls and places them side-by-side. If order matters, how many different ways can Xela do this?

   (A) 6 **(B)** 9 **(C)** 15 **(D)** 21 **(E)** None of these

Part B

11. If $2^{4x+2}4^{2x+4} = 8^{2x+8}$, what is the value of x?

   **(A)** 7 **(B)** $\dfrac{1}{2}$ **(C)** 3 **(D)** 0 **(E)** 4

12. A standard piano has 52 white keys and 36 black keys, for a total of 88 keys. In order to conform with the metric system, a new type of piano is made with 100 keys, with the ratio of white keys to black keys as close as possible to that of a standard piano. How many black keys will there be on the new piano?

   (A) 36 **(B)** 38 **(C)** 40 **(D)** 41 **(E)** 59

13. The number of local maxima of $f(x) = \sin x + \cos x$ with $x \in [0, 2\pi]$ is

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

14. Three gears are arranged in the configuration below. Gear A has 12 teeth, gear B has 8 teeth, and gear C has 15 teeth. Gear C is rotated three full rotations clockwise. What is the resulting position of gear A?

   A B C

   (A)   **(B)**   **(C)**   **(D)**   **(E)**
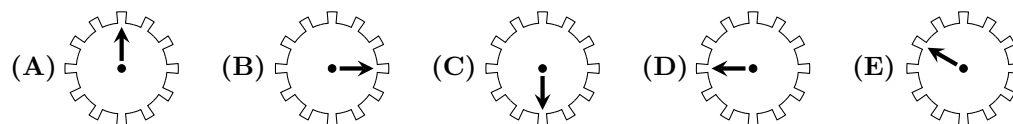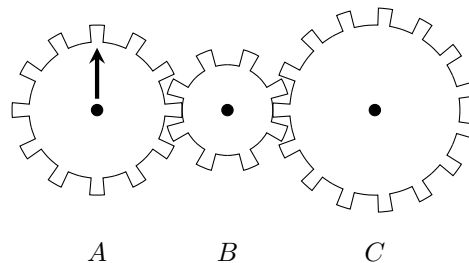
15. Eight teams are participating in a soccer tournament. Each team is meant to play each of the other teams the same number of times. Due to obscure tournament regulations, exactly twelve games must take place each day. What is the minimum number of days (of twelve games each) required for each of the teams to be scheduled to play each other the same number of times?

   (A) 3 **(B)** 7 **(C)** 8 **(D)** 12 **(E)** 14

16. A parallelogram $ABCD$ is constructed with the points $A(3,0)$, $B(0,q)$, $C(p,q)$, and $D(p+3,0)$. Diagonal AC lies along the line with equation $y = x - 3$. If the area of the parallelogram is 108, a possible value of $p + q$ is

   (A) -3 **(B)** 3 **(C)** 15 **(D)** 21 **(E)** 24

17. If $6066n$ is a perfect square and n is a positive integer, what is the smallest possible value of n?

   (A) 337   **(B)** 674   **(C)** 1011   **(D)** 2022   **(E)** 6066

18. The number of triangles with integer side lengths, positive area, and a perimeter of 20 is

   **(A)** 8   **(B)** 33   **(C)** 10   **(D)** 13   **(E)** 9

19. For three numbers x, y and z, such that $z \le y \le x$, we have the following:

$$\begin{aligned} x^2 - xy + y^2 &= z^2 \\ y^2 &= 2yz - xy \\ x &= 12 - z \end{aligned}$$

   What is the value of y?

   (A) 0   **(B)** 3   **(C)** 6   **(D)** 12   **(E)** 24

20. Below is a map of Equalador. Unlike in most cities, the blocks in Equalador are all equilateral triangles of the same size. A taxi wishes to get from point A to point B, and while there are many ways to make that trip, only some of those routes are the most efficient (that is, they take the least amount of driving). How many different routes can the taxi take, while ensuring the drive is as short as possible?
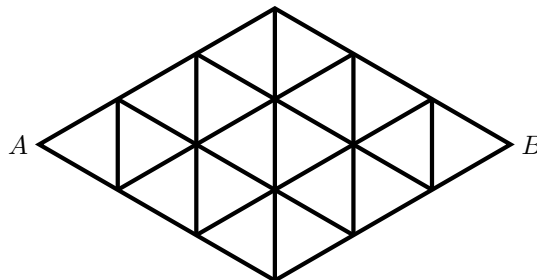
   (A) 2   **(B)** 10   **(C)** 20   **(D)** 40   **(E)** 120

Part C

21. In the diagram, quadrilateral $PQRS$ has $PQ = 39$, $QR = 36$, $RS = 9$, $SP = 12$, and $\angle PSR = 90°$. The area of quadrilateral $PQRS$ is
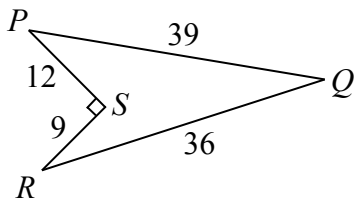
   (A) 216   **(B)** 270   **(C)** 324   **(D)** 392   **(E)** 432

22. The sequence shown begins with the term $x + 2022$ and each term after the first is 1 greater than the previous term.

$$(x + 2022), (x + 2023), (x + 2024), \ldots, (x + 6283), (x + 6284)$$

   For some values of the positive integer x, the sum of the terms in this sequence is a perfect square. What is the smallest such value of x?

   (A) 23   **(B)** 2022   **(C)** 394   **(D)** 6284   **(E)** 110

23. A fraction $\dfrac{a}{b}$ is *reduced* if a and b are integers such that $\gcd(a,b) = 1$. Let S be the set of all reduced fractions $\dfrac{a}{b}$ satisfying $1 \le a \le b \le 100$. How many elements of S have a terminating (i.e. non-repeating) decimal representation?

(A) 208   (B) 100   (C) 184   (D) 256   (E) None of these

24. The quadratic function $f(x) = x^2 - ax - 1$ satisfies $f(q) = f(2q^2) = f(3q^3)$ for at least one non-zero rational number q. What is the maximum possible value of a?

(A) $\dfrac{7}{8}$ (B) $\dfrac{14}{9}$ (C) 1   (D) $\dfrac{21}{10}$ (E) None of these

25. Alice and Bob take turns flipping a fair coin, with Alice going first. Let p denote the probability that Alice will flip two consecutive heads or two consecutive tails before Bob flips a head followed by a tail. The integer closest to $100p$ is

(A) 75   (B) 76   (C) 77   (D) 78   (E) 79