Special K

1: Determine all pairs of polynomials $(p(x), q(x))$ with complex coefficients such that

$$\begin{aligned} p(x^2) &= q(x)^2 \\ q(x^2) &= p(x)^2. \end{aligned}$$

We have $p(x^4) = q(x^2)^2 = p(x)^4$. Let a_i denote the coefficient of x^i in $p(x)$. If $a_0 \neq 0$, then we must have $a_1 = 0$ since $p(x^4)$ has no x-term, which then implies $a_2 = 0$ and so on. Hence $p(x) = 1$ or ζ_3 or ζ_3^2 in this case. If $a_0 = 0$, we may write $p(x) = r(x)x$ for some complex polynomial $q(x)$. Then $r(x)$ satisfies the same condition as $p(x)$. Hence $p(x) = cx^n$ for some nonnegative integer n and $c = 0, 1, \zeta_3, \zeta_3^2$.

2: Let N be a 2019-digit integer with no zero digits. Show that one can replace some (or none) but not all of the digits of N by 0 to obtain an integer divisible by 2019.

For $k = 0, 1, \ldots, 2018$, let a_k denote the number obtained by replacing the first k digits of N by 0. If any of the a_k is divisible by 2019, then we are done. Otherwise, there exists $k < l$ such that $a_k \equiv a_l \pmod{2019}$. Then 2019 divides $a_k - a_l$ which is obtained from N by replacing some digits by 0.

3: Does there exist a nonzero polynomial $p(x, y)$ in 2 variables with real coefficients such that for any real number a,

$$p(\lfloor a \rfloor, \lfloor a^2 \rfloor) = 0,$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to a?

The answer is no. Suppose for a contradiction that such a polynomial $p(x, y)$ exist. Then for any integer n,

$$p(n, n^2) = \cdots = p(n, n^2 + 2n) = 0.$$

As a polynomial in y, for any fixed n, $p(n, y)$ has absolutely bounded degrees. Hence for n large enough, $p(n, y)$ is the zero polynomial. This is clearly not possible since the leading coefficient of $p(x, y)$ when viewed as a polynomial in y is a fixed polynomial in x with finitely many roots.

4: Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x + y) + 1$ for all $x, y \in \mathbb{Q}$.

Setting $y = 1$ gives $f(x + 1) = f(x) + 1$ for all x. By induction, for any integer m, $f(m) = m + 1$ and $f(x + m) = f(x) + m$ for all x. Take any rational number n/m with $n, m \in \mathbb{Z}$. Then

$$f((n/m) \cdot m) = f(n/m)(m + 1) - (f(n/m) + m) + 1.$$

Solving gives $f(n/m) = n/m + 1$. Hence $f(x) = x + 1$ for all $x \in \mathbb{Q}$. It is easy to check that this function does satisfy the given formula.

5: Show that for any positive integer n, there exists a positive integer m, integers d_{ij} for $1 \le i \le m$ and $1 \le j \le n$ and rational numbers $c_1, \dots, c_m$ such that as polynomials in $x_1, \dots, x_n$,

$$\sum_{i=1}^{m} c_i \left(\sum_{j=1}^{n} d_{ij} x_j \right)^k = \begin{cases} x_1 x_2 \cdots x_n & \text{if } k = n, \\ 0 & \text{if } k = 1, \dots, n-1. \end{cases}$$

Solution 1. We denote each sum $\sum_{j=1}^{n} d_{ij} x_j$ by $F_i(x_1, \dots, x_n)$. We construct the data (c_i, F_i) by induction on n. When $n = 1$, we simply take $c_1 = 1$ and $F_1(x_1) = x_1$. Suppose now $(c_i, F_i(x_1, \dots, x_n))$ is given for $i = 1, \dots, m$. We construct the data for $n+1$. For $i = 1, \dots, m$, we set $G_i(x_1, \dots, x_{n+1}) = F_i(x_1, \dots, x_n) + x_{n+1}$. Then

$$\begin{aligned} \sum_{i=1}^{m} c_i G_i(x_1, \dots, x_{n+1})^k &= \sum_{j=0}^{k} \binom{k}{j} x_{n+1}^{k-j} \sum_{i=1}^{m} c_i F_i(x_1, \dots, x_n)^j \\ &= x_{n+1}^k \sum_{i=1}^{m} c_i + \sum_{i=1}^{m} c_i F_i(x_1, \dots, x_n)^k + H(x_1, \dots, x_{n+1}) \end{aligned}$$

where

$$H(x_1, \dots, x_{n+1}) = \begin{cases} (n+1) x_1 x_2 \cdots x_n x_{n+1} & \text{if } k = n+1, \\ 0 & \text{if } k = 1, \dots, n. \end{cases}$$

Hence, for $i = 1, \dots, m$, we set $d_i = c_i/(n+1)$. For $i = m+1, \dots, 2m$, we set $G_i(x_1, \dots, x_{n+1}) = F_{i-m}(x_1, \dots, x_n)$ and $d_i = -c_{i-m}/(n+1)$. Finally for $i = 2m+1$, we set $G_i = x_{n+1}$ and $d_i = -\sum_{j=1}^{m} c_j/(n+1)$. Then the data (d_i, G_i) for $i = 1, \dots, 2m+1$ does the job.

Solution 2. For every nonempty subset S of $\{1, 2, \dots, n\}$, let $F_S(x_1, \dots, x_n) = \sum_{i \in S} x_i$ and let $c_S = (-1)^{|S|}$. We compute the coefficient of each monomial in $\sum_S c_S F_S^k$. Consider first a monomial without one of the indeterminants, say, x_n. For any subset T of $\{1, 2, \dots, n-1\}$, the contribution from F_T^k and $F_{T \cup \{n\}}^k$ are equal but $c_T = -c_{T \cup \{n\}}$. Hence the total contribution is 0. If $k < n$, then there is not enough degree for every indeterminants to appear and so the sum is 0. If $k = n$, then only the term $x_1 x_2 \cdots x_n$ can appear and it can only show up when $S = \{1, 2, \dots, n\}$. The coefficient of $x_1 x_2 \cdots x_n$ in $(x_1 + \cdots + x_n)^n$ is $n!$. Therefore, replacing every c_S by $c_S/((-1)^n n!)$ does the job.

6: Given any two coprime positive integers a, b with $a < b$, one defines a Fibonacci sequence $\{F_n\}$ by

$$F_0 = a, \quad F_1 = b, \quad F_n = F_{n-1} + F_{n-2}, \forall n \ge 2.$$

Show that if $\{F_n\}$ is a Fibonacci sequence where for every prime p, there exists an index $n \ge 1$ such that p divides F_n, then a and b are consecutive terms in the standard Fibonacci sequence that starts with 0 and 1.

First note that the sequence that starts with $b - a$ and a is simply the given sequence shifted one term to the left and thus satisfies the same divisibility condition. If $b - a < a$, then we may conclude via induction. It remains to study the case $b - a \ge a$.

The key invariant property we need is

$$F_n^2 - F_{n-1} F_{n+1} = F_n^2 - F_{n-1}(F_n + F_{n-1}) = -(F_{n-1}^2 - F_{n-2} F_n).$$

Let $d = b^2 - a(a+b) = F_1^2 - F_0F_2$. Then for any $n \geq 2$, $F_n^2 - F_{n-1}F_{n+1} = \pm d$. Suppose for a contradiction that d has a prime divisor p. By assumption, $p \mid F_n$ for some $n \geq 1$. Then $p \mid F_{n-1}F_{n+1}$ and so $p \mid F_{n-1}$ or $p \mid F_{n+1}$. In either case, p divides two consecutive terms in the sequence and so p divides every term in sequence, contradicting the coprimeness of F_0 and F_1. Thus, $d = \pm 1$. It is then easy to see the only solution when $b - a \geq a$ is $a = 1$ and $b = 2$, which are two consecutive terms in the standard Fibonacci sequence.

Big E

1: Let N be a 2019-digit integer with no zero digits. Show that one can replace some (or none) but not all of the digits of N by 0 to obtain an integer divisible by 2019.

For $k = 0, 1, \ldots, 2018$, let a_k denote the number obtained by replacing the first k digits of N by 0. If any of the a_k is divisible by 2019, then we are done. Otherwise, there exists $k < l$ such that $a_k \equiv a_l \pmod{2019}$. Then 2019 divides $a_k - a_l$ which is obtained from N by replacing some digits by 0.

2: Does there exist a nonzero polynomial $p(x, y)$ in 2 variables with real coefficients such that for any real number a,

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where $\lfloor a \rfloor$ is the greatest integer less than or equal to a?

The answer is no. Suppose for a contradiction that such a polynomial $p(x, y)$ exist. Then for any integer n,

$$p(n, n^2) = \cdots = p(n, n^2 + 2n) = 0.$$

As a polynomial in y, for any fixed n, $p(n, y)$ has absolutely bounded degrees. Hence for n large enough, $p(n, y)$ is the zero polynomial. This is clearly not possible since the leading coefficient of $p(x, y)$ when viewed as a polynomial in y is a fixed polynomial in x with finitely many roots.

3: Let $\mathbb{N}$ denote the set of all positive integers. Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$, $f(2) = 4$ and $f(f(m) + f(n)) = f(f(m)) + f(n)$ for all $m, n \in \mathbb{N}$.

Setting $m = 1$ gives $f(f(n)+2) = f(n)+4$. From this, one obtains via induction that $f(2k) = 2k+2$ for any $k \in \mathbb{N}$. Since f is injective, it sends odd integers bigger than 1 to positive odd integers. Suppose $\alpha > 1$ is odd with $f(\alpha) = \beta$. Taking $m = 1$ and $n = \alpha$ gives $f(\beta + 2) = \beta + 4$. Taking $m = \alpha$ and $n = 1$ gives $f(\beta + 2) = f(\beta) + 2$. Hence $f(\beta) = \beta + 2$ and the same induction can be used to show $f(\beta + 2k) = \beta + 2k + 2$ for all $k \geq 0$. Suppose now the odd integer $\alpha > 1$ is chosen so that $\beta = f(\alpha)$ is minimal. If $f(3) > \beta$ and then $f(3) = f(\beta + (f(3) - \beta - 2))$. Injectivity then gives $f(3) = 5$ and the above implies $f(n) = n + 2$ for all odd integers $n > 1$ and so $\beta = 5$, contradicting $f(3) > \beta$. Hence we must have $f(3) = \beta$. Since $f(\beta + 2k) = \beta + 2k + 2$ for all $k \geq 0$, we see there cannot be any more odd integers between 3 and β. In other words, $f(3) = 5$. Therefore, $f(1) = 2$ and $f(n) = n + 2$ for all $n \geq 2$.

To check this function works, note that $f(n) \geq 2$ for all $n \in \mathbb{N}$ and $f(m)+f(n) \geq 2$ for all $m, n \in \mathbb{N}$. Hence $f(f(m) + f(n)) = f(m) + f(n) + 2$ and $f(f(m)) + f(n) = f(m) + 2 + f(n)$.

4: Show that for any positive integer n, there exists a positive integer m, integers d_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$ and rational numbers $c_1, \ldots, c_m$ such that as polynomials in $x_1, \ldots, x_n$,

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$$\begin{aligned}\sum_{i=1}^{m} c_i G_i(x_1,\ldots,x_{n+1})^k &= \sum_{j=0}^{k}\binom{k}{j} x_{n+1}^{k-j}\sum_{i=1}^{m} c_i F_i(x_1,\ldots,x_n)^j \\ &= x_{n+1}^k \sum_{i=1}^{m} c_i + \sum_{i=1}^{m} c_i F_i(x_1,\ldots,x_n)^k + H(x_1,\ldots,x_{n+1})\end{aligned}$$

where

$$H(x_1,\ldots,x_{n+1}) = \begin{cases}(n+1)x_1x_2\cdots x_nx_{n+1} & \text{if } k = n+1,\\ 0 & \text{if } k = 1,\ldots,n.\end{cases}$$

Hence, for $i = 1,\ldots,m$, we set $d_i = c_i/(n+1)$. For $i = m+1,\ldots,2m$, we set $G_i(x_1,\ldots,x_{n+1}) = F_{i-m}(x_1,\ldots,x_n)$ and $d_i = -c_{i-m}/(n+1)$. Finally for $i = 2m+1$, we set $G_i = x_{n+1}$ and $d_i = -\sum_{j=1}^{m} c_j/(n+1)$. Then the data (d_i, G_i) for $i = 1,\ldots,2m+1$ does the job.

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5: Evaluate the sum $\displaystyle\sum_{n=0}^{\infty}\frac{2}{n!}\frac{1}{n^4+n^2+1}$.

Do the only thing sensible at each step and it works!

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{2}{n!}\frac{1}{n^4+n^2+1} &= 2+\sum_{n=1}^{\infty}\frac{1}{n!n}\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)\\
&= 2+1-\sum_{n=1}^{\infty}\frac{1}{n^2+n+1}\left(\frac{1}{n!n}-\frac{1}{(n+1)!(n+1)}\right)\\
&= 3-\sum_{n=1}^{\infty}\frac{1}{n^2+n+1}\frac{n^2+n+1}{n(n+1)(n+1)!}\\
&= 3-\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+1)!}\\
&= 3-\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
&= 3-\frac{1}{2}+\sum_{n=2}^{\infty}\frac{1}{n}\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right)\\
&= \frac{5}{2}+\sum_{n=2}^{\infty}\frac{1}{(n+1)!}\\
&= \frac{5}{2}+(e-1-1-\frac{1}{2})=e.
\end{aligned}$$

All the rearrangements are justified because all the series involved are absolutely convergent.

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