Special K

1: Determine all pairs of polynomials $(p(x), q(x))$ with complex coefficients such that

$$\begin{aligned} p(x^2) &= q(x)^2 \\ q(x^2) &= p(x)^2. \end{aligned}$$

2: Let N be a 2019-digit integer with no zero digits. Show that one can replace some (or none) but not all of the digits of N by 0 to obtain an integer divisible by 2019.

3: Does there exist a nonzero polynomial $p(x, y)$ in 2 variables with real coefficients such that for any real number a,

$$p(\lfloor a \rfloor, \lfloor a^2 \rfloor) = 0,$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to a?

4: Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and $f(xy) = f(x)f(y) - f(x + y) + 1$ for all $x, y \in \mathbb{Q}$.

5: Show that for any positive integer n, there exists a positive integer m, integers d_{ij} for $1 \le i \le m$ and $1 \le j \le n$ and rational numbers $c_1, \ldots, c_m$ such that as polynomials in $x_1, \ldots, x_n$,

$$\sum_{i=1}^{m} c_i \left(\sum_{j=1}^{n} d_{ij} x_j \right)^k = \begin{cases} x_1 x_2 \cdots x_n & \text{if } k = n, \\ 0 & \text{if } k = 1, \ldots, n-1. \end{cases}$$

6: Given any two coprime positive integers a, b with $a < b$, one defines a Fibonacci sequence $\{F_n\}$ by

$$F_0 = a, \quad F_1 = b, \quad F_n = F_{n-1} + F_{n-2}, \forall n \ge 2.$$

Show that if $\{F_n\}$ is a Fibonacci sequence where for every prime p, there exists an index $n \ge 1$ such that p divides F_n, then a and b are consecutive terms in the standard Fibonacci sequence that starts with 0 and 1.

Big E

1: Let N be a 2019-digit integer with no zero digits. Show that one can replace some (or none) but not all of the digits of N by 0 to obtain an integer divisible by 2019.

2: Does there exist a nonzero polynomial $p(x, y)$ in 2 variables with real coefficients such that for any real number a,

$$p(\lfloor a \rfloor, \lfloor a^2 \rfloor) = 0,$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to a?

3: Let $\mathbb{N}$ denote the set of all positive integers. Find all injective functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$, $f(2) = 4$ and $f(f(m) + f(n)) = f(f(m)) + f(n)$ for all $m, n \in \mathbb{N}$.

4: Show that for any positive integer n, there exists a positive integer m, integers d_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$ and rational numbers $c_1, \ldots, c_m$ such that as polynomials in $x_1, \ldots, x_n$,

$$\sum_{i=1}^{m} c_i \left(\sum_{j=1}^{n} d_{ij} x_j \right)^k = \begin{cases} x_1 x_2 \cdots x_n & \text{if } k = n, \\ 0 & \text{if } k = 1, \ldots, n-1. \end{cases}$$

5: Evaluate the sum $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n!} \frac{1}{n^4 + n^2 + 1}$.

6: Given any two coprime positive integers a, b with $a < b$, one defines a Fibonacci sequence $\{F_n\}$ by

$$F_0 = a, \quad F_1 = b, \quad F_n = F_{n-1} + F_{n-2}, \forall n \geq 2.$$

Show that if $\{F_n\}$ is a Fibonacci sequence where for every prime p, there exists an index $n \geq 1$ such that p divides F_n, then a and b are consecutive terms in the standard Fibonacci sequence that starts with 0 and 1.