

SPECIAL K
Saturday November 10, 2018
10:00 am - 1:00 pm

- 1:** Let C be the circle of radius 1 centred at $(0, 1)$. Let D be the circle of radius 2 centred at $(a, 2)$ where $a > 0$ and D is externally tangent to C . Let E be the circle of radius r centred at $(x, 0)$ where $0 < x < a$ and E is externally tangent to both C and D . Find the values of x and r .
- 2:** Let a_n be the n^{th} positive integer k such that $\lfloor \sqrt{k} \rfloor$ divides k . Find n such that $a_n = 600$.
- 3:** A Mersenne prime is a prime of the form $p = 2^k - 1$ for some positive integer k . For a positive integer n , let $\sigma(n)$ be the sum of the positive divisors of n . Show that $\sigma(n)$ is a power of 2 if and only if n is a product of distinct Mersenne primes.
- 4:** Let $\{a_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Show that there exists a sequence $\{c_n\}$ of positive real numbers with $\lim_{n \rightarrow \infty} c_n = \infty$ such that $\sum_{n=1}^{\infty} c_n a_n < \frac{1}{2}$.
- 5:** Find the minimum possible value of $f'(2)$ given that $f(x)$ is a polynomial with nonnegative real coefficients such that $f(1) = 1$ and $f(2) = 3$.
- 6:** Let $a_0 = a_1 = 1$ and let $a_{2n} = a_{n-1} + a_n$ and $a_{2n+1} = a_n$ for $n \geq 1$. Define $f : \mathbf{Z}^+ \rightarrow \mathbf{Q}^+$ by $f(n) = \frac{a_n}{a_{n-1}}$. Show that f is bijective.

BIG E
Saturday November 10, 2018
10:00 am - 1:00 pm

- 1:** Let C be the sphere of radius 1 centred at $(0, 1, 1)$. Let D be the sphere of radius 2 centred at $(a, 2, 2)$ where $a > 0$ and D is externally tangent to C . Let E be the sphere of radius r centred at (x, r, r) where $0 < x < a$ and E is externally tangent to both C and D . Find the values of x and r .
- 2:** Let a_n be the n^{th} positive integer k such that $\lfloor \sqrt[3]{k} \rfloor$ divides k . Find n such that $a_n = 600$.
- 3:** Define $f : (1, \infty) \rightarrow \mathbf{R}$ by $f(x) = \int_x^{x^2} \frac{dt}{\ln t}$. Find the range of f .
- 4:** Let p be a prime number, let \mathbf{Z}_p be the field of integers modulo p , and let $M_3(\mathbf{Z}_p)$ be the ring of 3×3 matrices with entries in \mathbf{Z}_p . Find the number of functions $F : \mathbf{Z} \rightarrow M_3(\mathbf{Z}_p)$ such that $F(k+l) = F(k) + F(l)$ and $F(kl) = F(k)F(l)$ for all $k, l \in \mathbf{Z}$.
- 5:** Let $a_0 = a_1 = 1$ and let $a_{2n} = a_{n-1} + a_n$ and $a_{2n+1} = a_n$ for $n \geq 1$. Define $f : \mathbf{Z}^+ \rightarrow \mathbf{Q}^+$ by $f(n) = \frac{a_n}{a_{n-1}}$. Show that f is bijective.
- 6:** Let $n \in \mathbf{Z}^+$ and let $N = \{1, 2, 3, \dots, n\}$. Let S be a set of subsets of N with the property that for all $A, B \subseteq N$, if $A \in S$ and $A \subseteq B$ then $B \in S$. Define $f : [0, 1] \rightarrow \mathbf{R}$ by $f(x) = \sum_{A \in S} x^{|A|} (1-x)^{|N \setminus A|}$. Show that f is nondecreasing.