Solutions to the Special K Problems, 2017

1: Solve $\sin\left(x+\frac{\pi}{3}\right)+\cos x=\frac{1}{2}$ for $x\in\mathbf{R}$.

Solution: For $x\in[0,2\pi]$ we have

$$\begin{aligned}\sin\left(x+\tfrac{\pi}{3}\right)+\cos x=\tfrac{1}{2} &\iff \sin x\cdot\tfrac{1}{2}+\cos x\cdot\tfrac{\sqrt{3}}{2}+\cos x=\tfrac{1}{2}\\ &\iff \sin x=1-(2+\sqrt{3})\cos x\\ &\implies \sin^2 x=1-2(2+\sqrt{3})\cos x+(2+\sqrt{3})^2\cos^2 x\\ &\iff 1-\cos^2 x=1-(4+2\sqrt{3})\cos x+(7+4\sqrt{3})\cos^2 x\\ &\iff (8+4\sqrt{3})\cos^2 x-(4+2\sqrt{3})\cos x=0\\ &\iff (4+2\sqrt{3})(\cos x)(2\cos x-1)=0\\ &\iff \cos x=0\ \text{ or }\ \cos x=\tfrac{1}{2}\\ &\iff x=\tfrac{\pi}{2},\tfrac{3\pi}{2},\tfrac{\pi}{3}\text{ or }\tfrac{5\pi}{3}\end{aligned}$$

We check each of these to determine whether it gives a solution, and we find the solutions are $x=\frac{\pi}{2},\frac{5\pi}{3}$. Thus for $x\in\mathbf{R}$, the solutions are $x=\frac{\pi}{2}+2k\pi$ or $\frac{5\pi}{3}+2k\pi$ for some $k\in\mathbf{Z}$.

2: Let $0<r<1$. Let $A(r)$ be the area of the region bounded by the line through $(0,0)$ and $(1,1-r)$, the line through $(0,r)$ and $(1,1)$, the line through $(0,1)$ and $(1-r,0)$, and the line through $(r,1)$ and $(1,0)$. Find the value of r such that $A(r)=\frac{1}{25}$.

Solution: Let $A=(0,0)$, $B=(1,0)$, $C=(1,1)$ and $D=(0,1)$. Let K, L, M and N be the 4 given lines (in the given order). By symmetry (under rotating the square $ABCD$ by $90°$) the region bounded by the 4 lines is a square. Let P, Q, R and S be the vertices, in counterclockwise order with P being the point of intersection of K and M, so that area of the square is $A(r)=\ell^2$ where $\ell|P-Q|$. Let H be the horizontal line through P and let S be the point of intersection of H with N. Let V be the vertical line through B and C and let T be the point of intersection of V with K. Let $\theta=\angle QPS=\angle BAT$. Then $\cos\theta=\frac{|Q-P|}{|S-P|}=\frac{\ell}{r}$ and $\cos\theta=\frac{|B-A|}{|T-A|}=\frac{1}{\sqrt{(1-r)^2+1^2}}$ so we have $\ell=\frac{r}{\sqrt{r^2-2r+2}}$. Since $r>0$ we have

$$\begin{aligned}A(r)=\tfrac{1}{25}\iff \ell^2=\tfrac{1}{25}\iff \tfrac{r^2}{r^2-2r+2}=25\iff 25r^2=2-2r+r^2\iff 24r^2+2r-2=0\\ \iff 12r^2+r-1=0\iff (4r-1)(3r+1)=0\iff r=\tfrac{1}{4}.\end{aligned}$$

3: Let $S=\{1,2,\cdots,n\}$. Find the number of sets $\{A,B\}$ with $A,B\subseteq S$ and $A\cap B\neq\emptyset$.

Solution: Given an ordered pair (A,B) with $A,B\subseteq S$, we associate the n-tuple $(a_1,a_2,\cdots,a_n)$ where, for each index k, $a_k=0$ if $k\in A\cap B$, $a_k=1$ if $k\in A\setminus B$, $a_k=2$ if $k\in B\setminus A$, and $k=3$ if $k\notin A\cup B$. This establishes a bijective correspondence between the set of ordered pairs (A,B) with $A,B\subseteq S$ and the set of n-tuples $(a_1,a_2,\cdots,a_n)$ with each $a_k\in\{0,1,2,3\}$. The pairs (A,B) with $A\cap B\neq\emptyset$ correspond to the n-tuples with $a_k=0$ for at least one index k. The number n-tuples $(a_1,\cdots,a_n)$ with each $a_k\in\{0,1,2,3\}$ is equal to 4^n, and the number of such n-tuples with $a_k\neq 0$ for all k is equal to 3^n, and so the number of such n-tuples with $a_k=0$ for at least one index k is equal to 4^n-3^n. It follows that the number of ordered pairs (A,B) with $A,B\subseteq S$ and $A\cap B\neq\emptyset$ is equal to 4^n-3^n.

When $A\neq B$, the two ordered pairs (A,B) and (B,A) both determine the same (unordered) set $\{A,B\}$, and when $A=B$ the ordered pair $(A,B)=(A,A)$ determines the (unordered) set $\{A,A\}=\{A\}$. Since there are 2^n-1 nonempty subsets $\emptyset\neq A\subseteq S$, there are 2^n-1 pairs (A,A) giving 2^n-1 sets $\{A,A\}=\{A\}$. For the remaining $(4^n-3^n)-(2^n-1)$ ordered pairs (A,B) we have $A\neq B$ and these pairs determine a total of $\frac{1}{2}(4^2-3^n-2^n+1)$ sets $\{A,B\}$. Thus the total number of sets $\{A,B\}$ with $A,B\subseteq S$ and $A\cap B\neq\emptyset$ is equal to $(2^n-1)+\frac{1}{2}(4^n-3^n-2^n+1)$, that is $\frac{1}{2}(4^n-3^n+2^n-1)$.

4: For $x \in \mathbf{R}$, let $\langle x \rangle = x - \lfloor x \rfloor$. For $1 \leq n \in \mathbf{Z}$, let $x_n = \left\langle \frac{n}{\sqrt{2}} \right\rangle$. Show that the sequence (x_n) has a decreasing subsequence (x_{n_k}) with $x_{n_k} \to 0$ as $k \to \infty$.

Solution: Let n_k and m_k be the positive integers such that $(2-\sqrt{2})^k = n_k - m_k\sqrt{2}$. Then $n_1 = 2$ and $m_1 = 1$, and for $k \geq 1$, since $(2-\sqrt{2})^{k+1} = (2-\sqrt{2})(2-\sqrt{2})^k = (2-\sqrt{2})(n_k - m_k\sqrt{2}) = (2n_k + 2m_k) - (n_k + 2m_k)\sqrt{2}$ it follows that $n_{k+1} = 2n_k + 2m_k$ and $m_{k+1} = n_k + 2m_k$. From the recursion formula, we see that the sequences (n_k) and (m_k) are both increasing. Since $0 < (2-\sqrt{2}) < 1$, we have $0 < (2-\sqrt{2})^k < 1$ for all k, that is $0 < n_k - m_k\sqrt{2} < 1$ and hence $0 < \frac{n_k}{\sqrt{2}} - m_k < 1$. It follows that for all k we have $\lfloor \frac{n_k}{\sqrt{2}} \rfloor = m_k$ and hence

$$x_{n_k} = \left\langle \frac{n_k}{\sqrt{2}} \right\rangle = \frac{n_k}{\sqrt{2}} - m_k = \frac{1}{\sqrt{2}}(n_k - m_m\sqrt{2}) = \frac{1}{\sqrt{2}}(2-\sqrt{2})^k.$$

Thus the subsequence (x_{n_k}) is decreasing with $\lim\limits_{k \to \infty} x_{n_k} = 0$.

5: Let R be a ring with identity. Let n be an integer with $n \geq 2$. Suppose that $x^n = x$ for all $x \in R$. Show that $x^{n-1}y = y\,x^{n-1}$ for all $x, y \in R$.

Solution: First note that for all $w \in \mathbf{R}$, if $w^2 = 0$ then $w = 0$ because

$$w^2 = 0 \Longrightarrow w^{n-2} \cdot w^2 = w^{n-2} \cdot 0 \Longrightarrow w^n = 0 \Longrightarrow w = 0.$$

Next, we claim that for all $u \in R$, if $u^2 = u$ then u commutes with every element $y \in R$. To prove this claim, let $u \in R$ and suppose that $u^2 = u$. Then for any $y \in R$ we have

$$(uy - uyu)^2 = uyuy - uyuyu + uyuuy - uyuuyu = uyuy - uyuyu - uyuy + uyuyu = 0 \text{ and}$$
$$(yu - uyu)^2 = yuyu - yuuyu - uyuyu + uyuuyu = yuyu - yuyu - uyuyu + uyuyu = 0.$$

It follows (from the fact that $w^2 = 0 \Longrightarrow w = 0$) that $uy - uyu = 0 = yu - uyu$. Adding uyu to both sides gives $uy = yu$ as required, proving the claim. Finally, note that when $x \in R$ we have

$$(x^{n-1})^2 = x^{2n-2} = x^n \cdot x^{n-2} = x \cdot x^{n-2} = x^{n-1}$$

hence x^{n-1} commutes with every element $y \in R$ (by our earlier claim).

6: Let n be a positive integer. Show that $\prod_{k=1}^{n} \sin \frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}$ and hence find the product of all the lengths of the sides and diagonals of a regular $2n$-gon inscribed in the unit circle.

Solution: For $\theta = \frac{k\pi}{2\pi}$ with $k \in \{1, 2, \cdots, n-1\}$, we have $(e^{i\theta})^{2n} = e^{i\,2n\theta} = e^{i\,k\pi} = (e^{i\pi})^k = (-1)^k$ and we have $(e^{i\theta})^{2n} = (\cos\theta + i\sin\theta)^{2n}$ and so

$$\begin{aligned}(-1)^k &= (\cos\theta + i\sin\theta)^{2n} \\ &= \binom{2n}{0}\cos^{2n}\theta + i\binom{2n}{1}\cos^{2n-1}\theta\sin\theta - \binom{2n}{2}\cos^{2n-2}\theta\sin^2\theta - i\binom{2n}{3}\cos^{2n-3}\theta\sin^3\theta + \cdots\end{aligned}$$

Equating imaginary parts, then dividing by $\cos\theta\sin\theta$ (which is nonzero), then setting $x = \sin^2\theta$ gives

$$\begin{aligned}0 &= \binom{2n}{1}\cos^{2n-1}\theta\sin\theta - \binom{2n}{3}\cos^{2n-3}\theta\sin^3\theta + \binom{2n}{5}\cos^{2n-5}\theta\sin^5\theta - \cdots \\ &= \binom{2n}{1}\cos^{2n-2}\theta - \binom{2n}{3}\cos^{2n-4}\theta\sin^2\theta + \binom{2n}{5}\cos^{2n-6}\theta\sin^4\theta - \cdots \\ &= \binom{2n}{1}(1-x)^{n-1} - \binom{2n}{3}(1-x)^{n-2}x + \binom{2n}{5}(1-x)^{n-3}x^2 - \cdots \\ &= (-1)^{n-1}\Big(\binom{2n}{1}(x-1)^{n-1} + \binom{2n}{3}(x-1)^{n-2}x + \binom{2n}{5}(x-1)^{n-3}x^2 + \cdots + \binom{2n}{2n-1}x^{n-1}\Big)\end{aligned}$$

Thus the $n-1$ distinct numbers $x = \sin^2\theta$, where $\theta = \frac{k\pi}{2n}$ with $1 \le k < n$, are the roots of the degree $n-1$ polynomial

$$f(x) = \binom{2n}{1}(x-1)^{n-1} + \binom{2n}{3}(x-1)^{n-2}x + \binom{2n}{5}(x-1)^{n-3}x^2 + \cdots \binom{2n}{2n-1}x^{n-1}.$$

The leading coefficient is $c_{n-1} = \binom{2n}{1} + \binom{2n}{3} + \binom{2n}{5} + \cdots + \binom{2n}{2n-1} = 2^{2n-1}$ and the constant coefficient is $c_0 = (-1)^{n-1}\binom{2n}{1} = (-1)^{n-1}2n$ and so the product of the roots is

$$\prod_{k=1}^{n-1} \sin^2 \tfrac{k\pi}{2n} = \tfrac{(-1)^{n-1}c_0}{c_n} = \tfrac{2n}{2^{2n-1}} = \tfrac{n}{2^{2n-2}}.$$

Since $\sin\frac{\pi}{2} = 1$ and $\sin\frac{k\pi}{2n} > 0$ for $1 \le k < n$ this gives $\prod_{k=1}^{n} \sin\frac{k\pi}{2n} = \prod_{k=1}^{n-1}\sin\frac{k\pi}{2n} = \sqrt{\frac{n}{2^{2n-2}}} = \frac{\sqrt{n}}{2^{n-1}}$, as required.

For the second part of the question, let us place the vertices of the $2n$-gon at the points $e^{i\,k\pi/n}$ with $0 \le k < 2n$. The product of all the lengths of the sides and diagonals with one endpoint at 1 is equal to

$$\begin{aligned}p &= \prod_{k=1}^{2n-1}\left|e^{i\,k\pi/n} - 1\right| = \prod_{k=1}^{n-1}\left|e^{i\,k\pi/n} - 1\right| \cdot 2 \cdot \prod_{k=n+1}^{2n-1}\left|e^{i\,k\pi/n} - 1\right| = 2 \cdot \prod_{k=1}^{n-1}\left|e^{i\,k\pi/n} - 1\right|^2 \\ &= 2\prod_{k=1}^{n-1}\left((\cos\tfrac{k\pi}{n} - 1)^2 + (\sin\tfrac{k\pi}{n})^2\right) = 2\prod_{k=1}^{n-1}\left(2 - 2\cos\tfrac{k\pi}{n}\right) = 2\prod_{k=1}^{n-1} 4\sin^2\tfrac{k\pi}{2n} \\ &= 2^{2n-1}\Big(\prod_{k=1}^{n-1}\sin\tfrac{k\pi}{2n}\Big)^2 2^{2n-1} \cdot \tfrac{n}{2n-2} = 2n.\end{aligned}$$

Since for each of the $2n$ vertices of the polygon, the product of all the lengths of the sides and diagonals with one endpoint at that vertex will also be equal to p, and since each side and diagonal has two endpoints, the product of all the lengths is

$$P = \sqrt{p^{2n}} = \sqrt{(2n)^{2n}} = (2n)^n.$$

Solutions to the Big E Problems, 2016

1: Evaluate the infinite product $\prod_{n=1}^{\infty}\left(\frac{1}{2^n}\right)^{1/3^n}$.

Solution: Let $P_\ell=\prod_{n=1}^{\infty}\left(\frac{1}{2^n}\right)^{1/3^n}$ and let $S_\ell=\ln(P_\ell)$. Then

$$S_\ell=\sum_{n=1}^{\ell}\frac{1}{3^n}\ln\left(\frac{1}{2^n}\right)=-\ln 2\sum_{n=1}^{\ell}\frac{n}{3^n}.$$

For $|x|<1$ we have $\frac{1}{1-x}=1+x+x^2+\cdots$. Differentiate to get $\frac{1}{(1-x)^2}=1+2x+3x^2+\cdots$, multiply by x to get $\frac{x}{(1-x)^2}=z+2x^2+3x^3+\cdots$, then put in $x=\frac{1}{3}$ to get $\sum_{n=1}^{\infty}\frac{n}{3^n}=\frac{\frac{1}{3}}{(1-\frac{1}{3})^2}=\frac{3}{4}$. Thus

$$S_\ell=-\ln 2\sum_{n=1}^{\ell}\frac{n}{3^n}\longrightarrow-\tfrac{3}{4}\ln 2\quad\text{as}\quad\ell\to\infty$$

and so $P_\ell=e^{S_\ell}\longrightarrow e^{-\frac{3}{4}\ln 2}=2^{-3/4}=\frac{1}{2}\sqrt[4]{2}$ as $\ell\to\infty$.

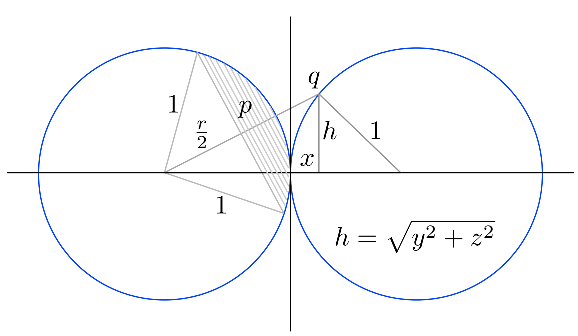
2: A point p is chosen at random on the surface of the sphere $(x-1)^2+y^2+z^2=1$ and a point q is chosen at random on the surface of the sphere $(x+1)^2+y^2+z^2=1$. Find the probability that the distance between p and q is at most 1.

Solution: Recall that on a sphere of radius R, a slice of thickness ℓ (that is the portion of the sphere which lies between two parallel planes separated by a distance of ℓ units) has area $2\pi R\,\ell$. Also recall that a spherical cap on the sphere is equal to a slice on the sphere which lies between two parallel planes, one of which is tangent to the sphere.

When $q=(x,y,z)$ lies on the second sphere so that we have $(x-1)^2+y^2=1$ (1), the distance r from q to the centre $(-1,0,0)$ of the first sphere is given by $(x+1)^2+y^2=r^2$ (2). Subtract (1) from (2) to get $4x=r^2-1$ and hence $4\,dx=2r\,dr$. When dr is small (infinitesimal), the set of points q which lie between r and dr units from $(-1,0,0)$ is a slice of the second sphere of thickness $dx=\frac{1}{2}r\,dr$ which has area $2\pi dx=\pi r\,dr$, and so the probability that q lies in this slice is equal to $\frac{\pi r\,dr}{4\pi}=\frac{1}{4}r\,dr$.

When $q=(x,y,z)$ is on the second sphere and r is the distance from q to $(-1,0,0)$, the set of points p on the first sphere which lie within 1 unit of q is a spherical cap which is equal to a slice of thickness $\ell=1-\frac{r}{2}$, which has area $2\pi\ell=2\pi\left(1-\frac{r}{2}\right)$, and so the probability that p lies within 1 unit of q is equal to $\frac{2\pi(1-\frac{r}{2})}{4\pi}=\frac{1}{2}\left(1-\frac{r}{2}\right)$ and we must have $1\le r\le 2$. Thus the required probability is

$$P=\int_{r=1}^{2}\tfrac{1}{2}\left(1-\tfrac{r}{2}\right)\cdot\tfrac{1}{4}r\;dr=\tfrac{1}{8}\int_{r=1}^{2}r-\tfrac{1}{2}r^2\;dr=\tfrac{1}{8}\left[\tfrac{1}{2}r^2-\tfrac{1}{6}r^3\right]_1^2=\tfrac{1}{8}\Big(\left(2-\tfrac{4}{3}\right)-\left(\tfrac{1}{2}-\tfrac{1}{6}\right)\Big)=\tfrac{1}{24}.$$



3: Let n and m be positive integers with $n < m$. Let $A_1, A_2, \cdots, A_m$ be nonempty subsets of $\{1, 2, \cdots, n\}$. Show that there exist nonempty disjoint subsets $I, J \subset \{1, 2, \cdots, m\}$ such that $\bigcup\limits_{i\in I} A_i = \bigcup\limits_{j\in J} A_j$.

Solution: To each set A_k we associate the vector $a_k = (a_{k,1}, a_{k,2}, \cdots, a_{k,n}) \in \mathbf{R}^n$ whose entries are given by $a_{k,\ell} = 1$ when $\ell \in A_k$ and $a_{k,\ell} = 0$ when $\ell \notin A_k$. Since $A_k \neq 0$ we have $a_k \neq 0$. Since $m > n$ the vectors $a_1, a_2, \cdots, a_m$ are linearly independent so we can choose $0 \neq t = (t_1, t_2, \cdots, t_m) \in \mathbf{R}^m$ so that $\sum\limits_{r=1}^{m} t_r a_r = 0$. Let $I = \{r \,|\, t_r > 0\}$ and let $J = \{r \,|\, t_r < 0\}$ so that we have

$$0 = \sum_{r=1}^{m} t_r a_r = \sum_{i\in I} t_i a_i + \sum_{j\in J} t_j a_j .$$

The sets I and J are clearly disjoint. We claim that I and J are nonempty. Choose an index k such that $t_k \neq 0$, say $t_k > 0$ (the case that $t_k < 0$ is similar). Since $t_k > 0$ we have $k \in I$ so that $I \neq \emptyset$. Since $A_k \neq \emptyset$ we can choose an index ℓ so that $a_{k,\ell} = 1$. Then, since each $t_i > 0$ and each $a_{i,\ell} \geq 0$, the ℓ^{th} entry of $\sum\limits_{i\in I} t_i a_i$ satisfies

$$\Big(\sum_{i\in I} t_i a_i\Big)_\ell = \sum_{i\in I} t_i a_{i,\ell} \geq t_k a_{k,\ell} = t_k > 0.$$

It follows that $J \neq \emptyset$, as claimed, because if we had $J = \emptyset$ then we would have $\sum\limits_{i\in I} t_i a_i = \sum\limits_{r=1}^{m} t_r a_r = 0$.

Finally note that $\bigcup\limits_{i\in I} A_i = \bigcup\limits_{j\in J} A_j$ because, as above, for each ℓ we have

$$\ell \in A_i \text{ for some } i \in I \iff a_{i,\ell} = 1 \text{ for some } i \in I \iff \Big(\sum_{i\in I} t_i a_i\Big)_\ell > 0$$
$$\iff \Big(\sum_{j\in J} t_j a_j\Big)_\ell < 0 \iff a_{j,\ell} = 1 \text{ for some } j \in J \iff \ell \in A_j \text{ for some } j \in J\,.$$

4: For $x \in \mathbf{R}$, let $\langle x \rangle = x - \lfloor x \rfloor$. For $n \in \mathbf{Z}^+$, let $S_n = \left\{k \in \mathbf{Z}^+ \middle| \left\langle \frac{n}{k} \right\rangle \geq \frac{1}{2}\right\}$. Find $\sum\limits_{k\in S_n} \varphi(k)$, where φ is the Euler phi function.

Solution: Recall that for $m \in \mathbf{Z}^+$ we have $\sum\limits_{d|m} \varphi(d) = m$ and note that for $m, k \in \mathbf{Z}^+$ we have $\left\lfloor \frac{m}{k} \right\rfloor = \sum\limits_{\ell \leq m\,,\, d|\ell} 1$. It follows that for $m, k \in \mathbf{Z}^+$ we have

$$\sum_{k\leq m} \varphi(k)\left\lfloor \tfrac{m}{k} \right\rfloor = \sum_{k\leq m}\Big(\varphi(k) \sum_{\ell\leq m\,,\, k|\ell} 1\Big) = \sum_{\ell\leq m}\sum_{k|\ell}\varphi(k) = \sum_{\ell\leq m} \ell = \tfrac{m(m+1)}{2}.$$

Note that for $m \in \mathbf{Z}^+$ and $x \in \mathbf{R}$, when $m \leq x < m + \frac{1}{2}$ so that $2m \leq 2x < 2m+1$ we have $\lfloor x \rfloor = m$ and $\lfloor 2x \rfloor = 2m$. and when $m + \frac{1}{2} \leq x < m+1$ so that $2m+1 \leq 2x < 2m+2$ we have $\lfloor x \rfloor = m$ and $\lfloor 2x \rfloor = 2m+1$. Thus for $x \in \mathbf{R}$ we have

$$\lfloor 2x \rfloor - 2\lfloor x \rfloor = \begin{cases} 0 \text{ if } 0 \leq \langle x \rangle < \frac{1}{2}, \\ 1 \text{ if } \frac{1}{2} \leq \langle x \rangle < 1. \end{cases}$$

Thus

$$\sum_{k\in S_n} \varphi(k) = \sum_{k\geq 1} \varphi(k)\Big(\left\lfloor \tfrac{2n}{k} \right\rfloor - 2\left\lfloor \tfrac{n}{k} \right\rfloor\Big) = \sum_{k\leq 2n} \varphi(k)\left\lfloor \tfrac{2n}{k} \right\rfloor - 2\sum_{k\leq n} \varphi(k)\left\lfloor \tfrac{n}{k} \right\rfloor = \tfrac{2n(2n+1)}{2} - 2\cdot\tfrac{n(n+1)}{2} = n^2.$$

5: Let R be a ring with identity. Let n be an integer with $n \geq 2$. Suppose that $x^n = x$ for all $x \in R$. Show that $x^{n-1}y = y\,x^{n-1}$ for all $x, y \in R$.

Solution: First note that for all $w \in \mathbf{R}$, if $w^2 = 0$ then $w = 0$ because

$$w^2 = 0 \Longrightarrow w^{n-2} \cdot w^2 = w^{n-2} \cdot 0 \Longrightarrow w^n = 0 \Longrightarrow w = 0.$$

Next, we claim that for all $u \in R$, if $u^2 = u$ then u commutes with every element $y \in R$. To prove this claim, let $u \in R$ and suppose that $u^2 = u$. Then for any $y \in R$ we have

$$(uy - uyu)^2 = uyuy - uyuyu + uyuuy - uyuuyu = uyuy - uyuyu - uyuy + uyuyu = 0 \text{ and}$$
$$(yu - uyu)^2 = yuyu - yuuyu - uyuyu + uyuuyu = yuyu - yuyu - uyuyu + uyuyu = 0.$$

It follows (from the fact that $w^2 = 0 \Longrightarrow w = 0$) that $uy - uyu = 0 = yu - uyu$. Adding uyu to both sides gives $uy = yu$ as required, proving the claim. Finally, note that when $x \in R$ we have

$$(x^{n-1})^2 = x^{2n-2} = x^n \cdot x^{n-2} = x \cdot x^{n-2} = x^{n-1}$$

hence x^{n-1} commutes with every element $y \in R$ (by our earlier claim).

6: Find $\displaystyle\lim_{n\to\infty} \int_0^\pi \frac{\sin x}{5 + 3\cos nx}\, dx$.

Solution: Let $f(x) = \sin x$ and $g(x) = \dfrac{1}{5 + 3\cos x}$ for $x \in \mathbf{R}$. Note that $\displaystyle\int_0^\pi f(x)\,dx = 2$ and for all $k \in \mathbf{Z}$

$$\int_{(k-1)\pi/n}^{k\pi/n} g(nx)\,dx = \tfrac{1}{n} \int_{(k-1)\pi}^{k\pi} g(x)\,dx = \tfrac{1}{n} \int_0^\pi g(x)\,dx.$$

For $n \in \mathbf{Z}^+$ let

$$I_n = \int_0^\pi \frac{\sin x}{5 + 3\cos(nx)}\,dx = \int_0^\pi f(x)\,g(nx)\,dx$$

and let $L_n(f)$ and $U_n(f)$ be the lower and upper Riemann sums for $f(x)$ on the partition of $[0, \pi]$ into n equal-sized subintervals, so we have $L_n(f) = \frac{\pi}{n} \sum\limits_{k=1}^{n} m_k$ and $U_n(f) = \frac{\pi}{n} \sum\limits_{k=1}^{n} M_k$ where $m_k = \min f(t)$ and $M_k = \max f(t)$ for $\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}$. Note that

$$\begin{aligned} I_n &= \int_0^\pi f(x)\,g(nx)\,dx = \sum_{k=1}^{n} \int_{(k-1)\pi/n}^{k\pi/n} f(x)\,g(nx)\,dx \\ &\leq \sum_{k=1}^{n} \int_{(k-1)\pi/n}^{k\pi/n} M_k\,g(nx)\,dx = \sum_{k=1}^{n} M_k \cdot \tfrac{1}{n} \int_0^\pi g(x)\,dx = \tfrac{1}{\pi}\,U_n(f) \int_0^\pi g(x)\,dx \end{aligned}$$

and similarly $I_n \geq \frac{1}{\pi}\,L_n(f) \int_0^\pi g(x)\,dx$. Thus we have

$$\tfrac{1}{\pi}\,L_n(f) \int_0^\pi g(x)\,dx \leq I_n \leq \tfrac{1}{\pi}\,U_n(f) \int_0^\pi g(x)\,dx$$

for all $n \in \mathbf{Z}^+$. Since $\lim\limits_{n\to\infty} L_n(f) = \lim\limits_{n\to\infty} U_n(f) = \int_0^\pi f(x)\,dx = 2$, it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} I_n = \tfrac{2}{\pi} \int_0^\pi g(x)\,dx = \tfrac{2}{\pi} \int_0^\pi \frac{dx}{5 + 3\cos x}\,.$$

Using the tangent half-angle substitution $\tan\frac{x}{2} = u$ so that $\cos x = \frac{1-u^2}{1+u^2}$ and $dx = \frac{2}{1+u^2}\,du$ we have

$$\int_{x=0}^\pi \frac{dx}{5 + 3\cos x} = \int_{u=0}^\infty \frac{\frac{2}{1+u^2}\,du}{5 + 3 \cdot \frac{1-u^2}{1+u^2}} = \int_0^\infty \frac{2\,du}{5(1+u^2) + 3(1-u^2)} = \int_0^\infty \frac{du}{4 + u^2} = \left[\tfrac{1}{2}\tan^{-1}\tfrac{u}{2}\right]_0^\infty = \tfrac{\pi}{4}$$

and so

$$\lim_{n\to\infty} I_n = \tfrac{2}{\pi} \cdot \tfrac{\pi}{4} = \tfrac{1}{2}.$$