SPECIAL K
Saturday November 5, 2017
10:00 am - 1:00 pm

1: Solve $\sin\left(x+\frac{\pi}{3}\right)+\cos x=\frac{1}{2}$ for $x \in \mathbf{R}$.

2: Let $0<r<1$. Let $A(r)$ be the area of the region bounded by the line through $(0,0)$ and $(1,1-r)$, the line through $(0,r)$ and $(1,1)$, the line through $(0,1)$ and $(1-r,0)$, and the line through $(r,1)$ and $(1,0)$. Find the value of r such that $A(r)=\frac{1}{25}$.

3: Let $S=\{1,2,\cdots,n\}$. Find the number of sets $\{A,B\}$ with $A,B\subseteq S$ and $A\cap B\neq\emptyset$.

4: For $x\in\mathbf{R}$, let $\langle x\rangle = x-\lfloor x\rfloor$. For $1\leq n\in\mathbf{Z}$, let $x_n=\left\langle\frac{n}{\sqrt{2}}\right\rangle$. Show that the sequence (x_n) has a decreasing subsequence (x_{n_k}) with $x_{n_k}\to 0$ as $k\to\infty$.

5: Let R be a ring with identity. Let n be an integer with $n\geq 2$. Suppose that $x^n=x$ for all $x\in R$. Show that $x^{n-1}y=y\,x^{n-1}$ for all $x,y\in R$.

6: Let n be a positive integer. Show that $\prod_{k=1}^{n}\sin\frac{k\pi}{2n}=\frac{\sqrt{n}}{2^{n-1}}$ and hence find the product of all the lengths of the sides and diagonals of a regular $2n$-gon inscribed in the unit circle.

BIG E
Saturday November 5, 2017
10:00 am - 1:00 pm

1: Evaluate the infinite product $\prod_{n=1}^{\infty} \left(\frac{1}{2^n}\right)^{1/3^n}$.

2: A point p is chosen at random on the surface of the sphere $(x-1)^2 + y^2 + z^2 = 1$ and a point q is chosen at random on the surface of the sphere $(x+1)^2 + y^2 + z^2 = 1$. Find the probability that the distance between p and q is at most 1.

3: Let n and m be positive integers with $n < m$. Let $A_1, A_2, \cdots, A_m$ be nonempty subsets of $\{1, 2, \cdots, n\}$. Show that there exist nonempty disjoint subsets $I, J \subset \{1, 2, \cdots, m\}$ such that $\bigcup_{i \in I} A_i = \bigcup_{j \in J} A_j$.

4: For $x \in \mathbf{R}$, let $\langle x \rangle = x - \lfloor x \rfloor$. For $n \in \mathbf{Z}^+$, let $S_n = \left\{ k \in \mathbf{Z}^+ \middle| \left\langle \frac{n}{k} \right\rangle \geq \frac{1}{2} \right\}$. Find $\sum_{k \in S_n} \varphi(k)$, where φ is the Euler phi function.

5: Let R be a ring with identity. Let n be an integer with $n \geq 2$. Suppose that $x^n = x$ for all $x \in R$. Show that $x^{n-1} y = y\, x^{n-1}$ for all $x, y \in R$.

6: Find $\lim_{n \to \infty} \int_0^{\pi} \frac{\sin x}{5 + 3 \cos nx}\, dx$.