Solutions to the Special K Problems, 2016

1: Let S be the circle of radius 1 centred at O, let T be the circle of radius 1 centred at Q and suppose S and T are tangent at P. A ray from O intersects S at the point A then intersects T at the points B and C. Suppose the distance from A to B is equal to the distance from B to C. Find the area of triangle APB.

Solution: Let $x = |AB|$ and let $\theta = \angle AOP$. By the Law of Cosines applied to the triangles OQB and OQC we have

$$\begin{aligned}\cos\theta = \frac{2^2+(x+1)^2-1^2}{2\cdot 2(x+1)} &= \frac{2^2+(2x+1)^2-1^2}{2\cdot 2(2x+1)}\\ (x^2+2x+4)(2x+1) &= (4x^2+4x+4)(x+1)\\ 2x^3+5x^2+10x+4 &= 4x^3+8x^2+8x+4\\ 2x^3+3x^2-2x &= 0\\ x(2x-1)(x+2) &= 0\end{aligned}$$

Since $x > 0$ we must have $x = \frac{1}{2}$. Note that triangle OQC is isoceles with $|OQ| = |OC| = 2$ and $|QC| = 1$. Using Pythagoras' Theorem to find the height h from vertex O, the area of triangle OQC is

$$|OQC| = \tfrac{1}{2}\,bh = \tfrac{1}{2}\cdot 1\cdot\sqrt{2^2-\left(\tfrac{1}{2}\right)^2} = \tfrac{1}{2}\cdot\tfrac{\sqrt{15}}{2} = \tfrac{\sqrt{15}}{4}.$$

Since $|AB| = \frac{1}{4}|OC|$ and $|OP| = \frac{1}{2}|OQ|$, the area of triangle APB is

$$|ABP| = \tfrac{1}{4}|OCP| = \tfrac{1}{8}|OQC| = \tfrac{\sqrt{15}}{32}.$$

2: Let m be a positive integer. Let $a_1 = m$ and let $a_{n+1} = \left\lfloor\sqrt{n\,a_n}\right\rfloor$ for $n \geq 1$. Show that there exists a positive integer N such that for all $n \geq N$ we have $a_n = n-3$.

Solution: Let $b_n = a_n - n + 3$ for $n \geq 1$. We need to show that there exists a positive integer N such that for all $n \geq N$ we have $b_n = 0$. Note that $a_1 = m \geq 1$, $a_2 = \lfloor\sqrt{1\cdot a_1}\rfloor \geq \lfloor\sqrt{1}\rfloor = 1$, $a_3 = \lfloor\sqrt{2\cdot a_2}\rfloor \geq \lfloor\sqrt{2}\rfloor = 1$ and $a_4 = \lfloor\sqrt{3\cdot a_3}\rfloor \geq \lfloor\sqrt{3}\rfloor = 1$ and so $b_1 \geq 3$, $b_2 \geq 2$, $b_3 \geq 1$ and $b_4 \geq 0$. Let $n \geq 4$ and suppose, inductively that $b_n \geq 0$, that is $a_n - n + 3 \geq 0$. Then $a_n \geq n-3$ so we have

$$na_n \geq n^2-3n \geq (n^2-3n)-(n-4) = n^2-4n+4 = (n-2)^2.$$

Thus $\sqrt{na_n} \geq n-2$ hence $a_{n+1} = \lfloor\sqrt{na_n}\rfloor \geq \lfloor n-2\rfloor = n-2$, and so $b_{n+1} = a_{n+1}-n+2 \geq (n-2)-n+2 = 0$. By induction, it follows that $b_n \geq 0$ for all $n \geq 1$ (with $b_n > 0$ for $n = 1,2,3$).

Next, we claim that if $b_n > 0$ then $b_{n+1} < b_n$. Let $n \geq 1$ and suppose that $b_n > 0$, that is $a_n - n + 3 > 0$. Then $n < a_n + 3$ so that $n \leq a_n + 2$ and so $na_n \leq {a_n}^2 + 2a_n < (a_n+1)^2$ hence $\sqrt{na_n} < a_n + 1$. Thus $a_{n+1} = \lfloor\sqrt{na_n}\rfloor \leq \sqrt{na_n} < a_n + 1$. It follows that $b_{n+1} = a_{n+1} - n + 2 < (a_n+1) - n + 2 = a_n - n + 3 = b_n$. Thus the sequence $\{b_n\}$ is strictly decreasing for as long as $b_n > 0$. It follows that there exists a positive integer N such that $b_N \leq 0$. Since $b_n \geq 0$ for all n, we must have $b_N = 0$. We also remark that since $b_1, b_2, b_3 > 0$ we must have $N \geq 4$.

Let $n \geq N \geq 4$ and suppose, inductively, that $b_n = 0$, that is $a_n = n-3$. Then $na_n = n(n-3) = n^2-3n$. Notice that

$$(n-2)^2 = (n^2-3n)-(n-4) \leq n^2-3n < (n^2-3n)+(n+1) = (n-1)^2,$$

so we have $(n-2)^2 \leq na_n < (n-1)^2$. Thus $n-2 \leq \sqrt{na_n} < n-1$, so $a_{n+1} = \lfloor\sqrt{na_n}\rfloor = n-2$, hence $b_{n+1} = a_{n+1} - n + 2 = 0$. By induction, it follows that $b_n = 0$ for all $n \geq N$, as required.

3: For positive integers n and k, let $\sigma(n,k)$ be the sum of all the divisors d of n with $\frac{n}{k} \leq d \leq k$. Find $S_k = \sum_{n=1}^{k^2} \sigma(n,k)$.

Solution: For each $d \leq k$, the number d occurs as a term in the sum $\sigma(n,k)$ when $d|n$ and $n \leq kd$, that is when $n \in \{1 \cdot d, 2 \cdot d, 3 \cdot d, \cdots, k \cdot d\}$. Thus each $d \leq k$ occurs exactly k times in the double sum S_k, and so

$$S_k = \sum_{d=1}^{k} kd = k(1+2+3+\cdots+k) = \tfrac{k^2(k+1)}{2}.$$

4: A game begins with a pile of n coins. Players A and B take turns with A going first. At each turn a player removes a nonzero perfect square number of coins from the pile. The player who removes the last coin wins. Show that there are infinitely many values of n for which player B has a winning strategy.

Solution: Let W be the set of all $n \geq 0$ such that the player whose turn it is wins when presented with a pile of n coins, let L be the set of $n \geq 0$ such that the player whose turn it is loses when presented with a pile of n coins, and let S be the set of perfect squares $S = \{1, 4, 9, \cdots\}$. Note that $W \neq \emptyset$; indeed $0 \in L$ (since a player who is presented with 0 coins has just lost the game) and $2 \in L$ (since given a pile of 2 coins the player is forced to remove 1 coin and lose). Also note that $W \cup L = \mathbf{N}$ by induction; indeed given $n \in \mathbf{N}$, if we suppose that every $k < n$ lies either in W or in L then it follows that n lies in W when there exists $s \in S$ with $s \leq n$ such that $n - s \in L$ and that n lies in L when no such $s \in S$ exists.

Suppose, for a contradiction, that L is finite. Let $m = \max(L)$ so that for all $n > m$ we have $n \in W$. Consider the number $n = m^2 + m + 1$. Note that $n > m$ so we have $n \in W$. For $k \geq m+1$ we have $n - k^2 \leq n - (m+1)^2 = (m^2+m+1) - (m^2+2m+1) = -m < 0$ so when $k > m$ we cannot remover $s = k^2$ coins. For $k \leq m$ we have $n - k^2 \geq n - m^2 = (m^2+m+1) - m^2 = m+1 > m$ so that $n - k^2 \in W$. Thus when $k > m$ the player cannot remove $s = k^2$ coins and when $k \leq m$ and $s = k^2$ we have $s - k^2 > m$ so that $s - k^2 \in W$ and so when the player removes s coins their opponent wins. But this shows that $n \in L$ giving the desired contradiction.

5: Let $f : [0,1] \to \mathbf{R}$ with $f(0) > 0$ and $f(1) < 0$. Suppose that there exists a continuous function $g : [0,1] \to \mathbf{R}$ such that $f + g$ is increasing. Show that there exists $c \in (0,1)$ such that $f(c) = 0$.

Solution: Choose a continuous function $g : [0,1] \to \mathbf{R}$ such that the function $h = f + g$ is increasing. Let $P = \{x \in [0,1] \,|\, f(x) \geq 0\}$ and let $c = \sup P$. We claim that $f(c) = 0$. Let $x \in P$. Since $x \in P$ we have $f(x) \geq 0$. Since $x \in P$ and $c = \sup P$ we have $x \leq c$. Since $x \leq c$ and h is increasing we have $h(x) \leq h(c)$. Thus $g(x) \leq f(x) + g(x) = h(x) \leq h(c)$. Since g is continuous with $g(x) \leq h(c)$ for all $x \in P$ and $c = \sup P$, it follows that $g(c) \leq h(c)$. Since $g(c) \leq h(c) = f(c) + g(c)$ we have $f(c) \geq 0$.

Since $f(c) \geq 0$ we have $g(c) \leq f(c) + g(c) = h(c)$. Since h is increasing and $c \leq 1$ we have $h(c) \leq h(1)$. Since $f(1) < 0$ we have $h(1) = f(1) + g(1) < g(1)$. Thus $g(c) \leq h(c) \leq h(1) < g(1)$. By the Intermediate Value Theorem, we can choose t with $c \leq t \leq 1$ such that $g(t) = h(c)$. Since h is increasing and $c \leq t$ we have $h(c) \leq h(t)$. Thus $g(t) = h(c) \leq h(t) = f(t) + g(t)$ and so $f(t) \geq 0$. Since $f(t) \geq 0$ we have $t \in P$. Since $t \in P$ and $c \leq t$ and $c = \sup P$ we have $c = t$. Thus $g(c) = g(t) = h(c) = f(c) + g(c)$ and so $f(c) = 0$, as claimed. Finally, note that since $f(0) \neq 0$ and $f(1) \neq 0$ and $f(c) = 0$, we cannot have $f(c) \in \{0,1\}$ so we have $c \in (0,1)$.

6: Let m be a positive integer. Let S be a set of m-element subsets of $\mathbf{Z}$. Suppose that for all $A, B \in S$ we have $A \cap B \neq \emptyset$. Show that there exists a finite set $F \subseteq \mathbf{Z}$ such that for all $A, B \in S$ we have $A \cap B \cap F \neq \emptyset$.

Solution: Consider the following statement about positive integers m and n: for all sets S of m-element sets of integers and all sets T of n-element sets of integers, if $A \cap B \neq \emptyset$ for all $A \in S$ and $B \in T$ then there exists a finite set of integers F such that $A \cap B \cap F \neq \emptyset$ for all $A \in S$ and $B \in T$. If we can prove that this statement is true for all positive integers m and n then the problem follows by taking $n = m$ and $T = S$.

When $m = n = 1$, this statement is clearly true (indeed if $S = \emptyset$ or $T = \emptyset$ we can take $F = \emptyset$, and if $S \neq \emptyset$ and $T \neq \emptyset$ with say $\{a\} \in S$ and $\{b\} \in T$, then the condition that $A \cap B \neq \emptyset$ for all $A \in S$ and $B \in T$ forces $a = b$ and $S = T = \big\{\{a\}\big\}$ so we can take $F = \{a\}$).

Let $m, n \geq 2$ and suppose, inductively, that for all $k < m$ and $\ell < n$, for all sets U of k-element sets of integers, and all sets V of ℓ-element sets of integers, if $X \cap Y \neq \emptyset$ for all $X \in U$ and $Y \in V$ then there exists a finite set of integers G such that $X \cap Y \cap G \neq \emptyset$ for all $X \in U$ and $Y \in V$. Let S be a set of m-element sets and let T be a set of n-element sets, and suppose that $A \cap B \neq \emptyset$ for all $A \in S$ and $B \in T$. We need to show that there exists a finite set of integers F such that $A \cap B \cap F \neq \emptyset$ for all $A \in S$ and $B \in T$.

If $S = \emptyset$ or $T = \emptyset$ then we can let $F = \emptyset$ and then, vacuously, we have $A \cap B \cap F \neq \emptyset$ for all $A \in S$ and $B \in T$. Suppose that $S \neq \emptyset$ and $T \neq \emptyset$. Choose $P \in S$ and $Q \in T$. For subsets $C, D \subseteq P \cup Q$, let $S_C = \big\{A \in S \big| A \cap (P \cup Q) = C\big\}$ and $T_D = \big\{B \in T \big| B \cap (P \cup Q) = D\big\}$. Note that since every $A \in S$ has a nonempty intersection with Q, hence with $P \cup Q$, if follows that S is the union of the sets S_C with $\emptyset \neq C \subseteq P \cup Q$. Similarly, T is the union of the sets T_D with $\emptyset \neq D \subseteq P \cup Q$. For each pair of nonempty subsets $C, D \subseteq P \cup Q$ we shall construct a finite set of integers $F_{C,D}$ such that $A \cap B \cap F_{C,D} \neq \emptyset$ for all $A \in S_C$ and $B \in T_D$. Once we have constructed these sets $F_{C,D}$ we can let F be the union of all these sets $F_{C,D}$, and then we have $A \cap B \cap F \neq \emptyset$ for all $A \in S$ and $B \in T$, so the proof will be complete.

If $C \cap D \neq \emptyset$, we define $F_{C,D} = C \cap D$. Then for $A \in S_C$ we have $A \cap (P \cup Q) = C$ so $C \subseteq A$ hence $C \cap D \subseteq A$ Similarly, for $B \in T_D$ we have $C \cap D \subseteq B$. Thus when $A \in S_C$ and $B \in T_D$ we have $C \cap D \subseteq A \cap B$ so that $A \cap B \cap F_{C,D} = (A \cap B) \cap (C \cap D) = C \cap D \neq \emptyset$.

Suppose that $C \cap D = \emptyset$. Note that for $A \in S_C$ and $B \in T_D$, the sets A and B intersect outside $P \cup Q$. Let $U = \big\{A \setminus C \big| A \in S_C\big\}$ and $V = \big\{B \setminus D \big| B \in T_D\big\}$. Note that U is a set of k-element sets where $k = m - |C| < m$ and V is a set of ℓ-element sets where $\ell = n - |D| < n$. Note that for $A \in S_C$ and $B \in T_D$, if we let $X = A \setminus C \in U$ and $Y = B \setminus D \in V$ then we have $X \cap Y = (A \setminus C) \cap (B \setminus D) = A \cap B \neq \emptyset$. By the induction hypothesis, we can choose a finite set of integers $F_{C,D}$ so that $X \cap Y \cap F_{C,D} \neq \emptyset$ for all $X \in U$ and $Y \in V$. Then for all $A \in S_C$ and $B \in T_D$ we have $A \cap B \cap F_{C,D} = X \cap Y \cap F_{C,D} \neq \emptyset$.

Solutions to the Big E Problems, 2016

1: Define $f : \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x,y) = \big(4x - 3y + 1\,,\, 2x - y + 1\big)$. Find $f^n(2,1)$ where n is a positive integer and f^n is defined recursively by $f^1 = f$ and $f^k = f \circ f^{k-1}$.

Solution: Let $(x_0, y_0) = (2,1)$ and let

$$(x_n, y_n) = f^n(2,1) = f\big(f^{n-1}(2,1)\big) = f(x_{n-1}, y_{n-1}) = \big(4x_{n-1} - 3y_{n-1} + 1\,,\, 2x_{n-1} - y_{n-1} + 1\big)$$

for $n \geq 1$. Let $a_n = x_n - y_n$ and $b_n = 2x_n - 3y_n$ for $n \geq 0$. Then $a_0 = 1$ and $b_0 = 1$ and

$$\begin{aligned} a_n &= (4x_{n-1} - 3y_{n-1} + 1) - (2x_{n-1} - y_{n-1} + 1) = 2x_{n-1} - 2y_{n-1} = 2a_{n-1} \\ b_n &= 2(4x_{n-1} - 3y_{n-1} + 1) - 3(2x_{n-1} - y_{n-1} + 1) = 2x_{n-1} - 2y_{n-1} = 2a_{n-1} - 3y_{n-1} - 1 = b_{n-1} - 1 \end{aligned}$$

for all $n \geq 1$. Since $a_0 = 1$ and $a_n = 2a_{n-1}$ we have $a_n = 2^n$ for all $n \geq 0$. Since $b_0 = 1$ and $b_n = b_{n-1} - 1$ we have $b_n = 1 - n$ for all $n \geq 0$. Since $x_n - y_n = a_n = 2^n$ and $2x_n - 3x_n = b_n = 1 - n$, we have $x_n = 3\cdot 2^n + n - 1$ and $y_n = 2\cdot 2^n + n - 1$ for all $n \geq 0$. Thus $f^n(2,1) = (x_n, y_n) = \big(3\cdot 2^n + n - 1\,,\, 2\cdot 2^n + n - 1\big)$ for all $n \geq 1$.

2: Let p_k denote the k^{th} prime number. Show that $\displaystyle\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{{p_k}^n} < \frac{3}{2} - \ln 2$.

Solution: Recall that for $|x| < 1$ we have $\ln(1+x) = x - \frac{1}{2}^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$. By Abel's Theorem we can also put in $x = 1$ to get $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

Since $p_1 = 2$ and $p_k = 2k - 1$ for $k = 2, 3, 4$ and $p_k > 2k - 1$ for $k > 4$, we have

$$\begin{aligned} \sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{{p_k}^n} &= \sum_{k=1}^{\infty}\sum_{n=2}^{\infty}\frac{1}{{p_k}^n} < \sum_{n=2}^{\infty}\frac{1}{2^n} + \sum_{k=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{(2k-1)^n} = \frac{1}{4}\cdot\frac{1}{1-\frac{1}{2}} + \sum_{k=2}^{\infty}\frac{1}{(2k-1)^2}\cdot\frac{1}{1-\frac{1}{2k-1}} \\ &= \frac{1}{2} + \sum_{k=2}^{\infty}\frac{1}{(2k-1)(2k-2)} = \frac{1}{2} + \sum_{k=2}^{\infty}\left(\frac{1}{2k-2} - \frac{1}{2k-1}\right) = \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots \\ &= \frac{3}{2} - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = \frac{3}{2} - \ln 2. \end{aligned}$$

3: A game begins with a pile of n coins. Players A and B take turns with A going first. At each turn a player removes a nonzero perfect square number of coins from the pile. The player who removes the last coin wins. Show that there are infinitely many values of n for which player B has a winning strategy.

Solution: Let W be the set of all $n \geq 0$ such that the player whose turn it is wins when presented with a pile of n coins, let L be the set of $n \geq 0$ such that the player whose turn it is loses when presented with a pile of n coins, and let S be the set of perfect squares $S = \{1, 4, 9, \cdots\}$. Note that $W \neq \emptyset$; indeed $0 \in L$ (since a player who is presented with 0 coins has just lost the game) and $2 \in L$ (since given a pile of 2 coins the player is forced to remove 1 coin and lose). Also note that $W \cup L = \mathbf{N}$ by induction; indeed given $n \in \mathbf{N}$, if we suppose that every $k < n$ lies either in W or in L then it follows that n lies in W when there exists $s \in S$ with $s \leq n$ such that $n - s \in L$ and that n lies in L when no such $s \in S$ exists.

Suppose, for a contradiction, that L is finite. Let $m = \max(L)$ so that for all $n > m$ we have $n \in W$. Consider the number $n = m^2 + m + 1$. Note that $n > m$ so we have $n \in W$. For $k \geq m + 1$ we have $n - k^2 \leq n - (m+1)^2 = (m^2 + m + 1) - (m^2 + 2m + 1) = -m < 0$ so when $k > m$ we cannot remover $s = k^2$ coins. For $k \leq m$ we have $n - k^2 \geq n - m^2 = (m^2 + m + 1) - m^2 = m + 1 > m$ so that $n - k^2 \in W$. Thus when $k > m$ the player cannot remove $s = k^2$ coins and when $k \leq m$ and $s = k^2$ we have $s - k^2 > m$ so that $s - k^2 \in W$ and so when the player removes s coins their opponent wins. But this shows that $n \in L$ giving the desired contradiction.

4: Let n be a positive integer. Find the largest integer m such that there exist m subsets $S_1, S_2, \cdots, S_m \subseteq \{1,2,3,\cdots,n\}$ with the property that each set S_k has an odd number of elements and each set $S_k \cap S_l$ with $k \neq l$ has an even number of elements.

Solution: We claim that the largest possible value is $m = n$. Note that there do exist n such sets, namely the sets $S_k = \{k\}$. It remains to show that there can be at most n such sets. Let $m \in \mathbf{Z}^+$ and suppose that we have such sets $S_1, S_2, \cdots, S_m$. For each index k, let u_k be the column vector $u_k = (u_{k,1}, u_{k,2}, \cdots, u_{k,n})^T \in {\mathbf{Z}_2}^n$ with $u_{k,j} = 1$ if $j \in S_k$ and $u_{k,j} = 0$ if $j \neq S_k$. Let $A = \big(u_1, u_2, \cdots, u_n\big) \in M_{n\times m}(\mathbf{Z}_2)$. Since each S_k has an odd number of elements, we have ${u_k}^T u_k = 1$ for all indices k. Since each $S_k \cap S_\ell$ has an even number of elements, we have ${u_k}^T u_\ell = 0$ for all $k \neq \ell$. It follows that $A^T A = I$. Let $t = (t_1, t_2, \cdots, t_m)^T \in {\mathbf{Z}_2}^m$ and suppose that $\sum t_i u_i = 0$. Then we have $At = \sum t_i u_i = 0$ and so $t = It = A^T At = 0$. This shows that $\{u_1, u_2, \cdots, u_m\}$ is linearly independent, and so $m \leq n$, as required.

5: Let m be a positive integer. Let S be a set of m-element subsets of $\mathbf{Z}$. Suppose that for all $A, B \in S$ we have $A \cap B \neq \emptyset$. Show that there exists a finite set $F \subseteq \mathbf{Z}$ such that for all $A, B \in S$ we have $A \cap B \cap F \neq \emptyset$.

Solution: Consider the following statement about positive integers m and n: for all sets S of m-element sets of integers and all sets T of n-element sets of integers, if $A \cap B \neq \emptyset$ for all $A \in S$ and $B \in T$ then there exists a finite set of integers F such that $A \cap B \cap F \neq \emptyset$ for all $A \in S$ and $B \in T$. If we can prove that this statement is true for all positive integers m and n then the problem follows by taking $n = m$ and $T = S$.

When $m = n = 1$, this statement is clearly true (indeed if $S = \emptyset$ or $T = \emptyset$ we can take $F = \emptyset$, and if $S \neq \emptyset$ and $T \neq \emptyset$ with say $\{a\} \in S$ and $\{b\} \in T$, then the condition that $A \cap B \neq \emptyset$ for all $A \in S$ and $B \in T$ forces $a = b$ and $S = T = \big\{\{a\}\big\}$ so we can take $F = \{a\}$).

Let $m, n \geq 2$ and suppose, inductively, that for all $k < m$ and $\ell < n$, for all sets U of k-element sets of integers, and all sets V of ℓ-element sets of integers, if $X \cap Y \neq \emptyset$ for all $X \in U$ and $Y \in V$ then there exists a finite set of integers G such that $X \cap Y \cap G \neq \emptyset$ for all $X \in U$ and $Y \in V$. Let S be a set of m-element sets and let T be a set of n-element sets, and suppose that $A \cap B \neq \emptyset$ for all $A \in S$ and $B \in T$. We need to show that there exists a finite set of integers F such that $A \cap B \cap F \neq \emptyset$ for all $A \in S$ and $B \in T$.

If $S = \emptyset$ or $T = \emptyset$ then we can let $F = \emptyset$ and then, vacuously, we have $A \cap B \cap F \neq \emptyset$ for all $A \in S$ and $B \in T$. Suppose that $S \neq \emptyset$ and $T \neq \emptyset$. Choose $P \in S$ and $Q \in T$. For subsets $C, D \subseteq P \cup Q$, let $S_C = \big\{A \in S \big| A \cap (P \cup Q) = C\big\}$ and $T_D = \big\{B \in T \big| B \cap (P \cup Q) = D\big\}$. Note that since every $A \in S$ has a nonempty intersection with Q, hence with $P \cup Q$, if follows that S is the union of the sets S_C with $\emptyset \neq C \subseteq P \cup Q$. Similarly, T is the union of the sets T_D with $\emptyset \neq D \subseteq P \cup Q$. For each pair of nonempty subsets $C, D \subseteq P \cup Q$ we shall construct a finite set of integers $F_{C,D}$ such that $A \cap B \cap F_{C,D} \neq \emptyset$ for all $A \in S_C$ and $B \in T_D$. Once we have constructed these sets $F_{C,D}$ we can let F be the union of all these sets $F_{C,D}$, and then we have $A \cap B \cap F \neq \emptyset$ for all $A \in S$ and $B \in T$, so the proof will be complete.

If $C \cap D \neq \emptyset$, we define $F_{C,D} = C \cap D$. Then for $A \in S_C$ we have $A \cap (P \cup Q) = C$ so $C \subseteq A$ hence $C \cap D \subseteq A$ Similarly, for $B \in T_D$ we have $C \cap D \subseteq B$. Thus when $A \in S_C$ and $B \in T_D$ we have $C \cap D \subseteq A \cap B$ so that $A \cap B \cap F_{C,D} = (A \cap B) \cap (C \cap D) = C \cap D \neq \emptyset$.

Suppose that $C \cap D = \emptyset$. Note that for $A \in S_C$ and $B \in T_D$, the sets A and B intersect outside $P \cup Q$. Let $U = \big\{A \setminus C \big| A \in S_C\big\}$ and $V = \big\{B \setminus D \big| B \in T_D\big\}$. Note that U is a set of k-element sets where $k = m - |C| < m$ and V is a set of ℓ-element sets where $\ell = n - |D| < n$. Note that for $A \in S_C$ and $B \in T_D$, if we let $X = A \setminus C \in U$ and $Y = B \setminus D \in V$ then we have $X \cap Y = (A \setminus C) \cap (B \setminus D) = A \cap B \neq \emptyset$. By the induction hypothesis, we can choose a finite set of integers $F_{C,D}$ so that $X \cap Y \cap F_{C,D} \neq \emptyset$ for all $X \in U$ and $Y \in V$. Then for all $A \in S_C$ and $B \in T_D$ we have $A \cap B \cap F_{C,D} = X \cap Y \cap F_{C,D} \neq \emptyset$.

6: Let $f:\mathbf{R}\to\mathbf{R}$ be continuous with $f(x)\geq 0$ for all $x\in\mathbf{R}$ and suppose $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $r>0$ and $n\in\mathbf{Z}^+$, let $B_n(r)=\big\{x\in\mathbf{R}^n\big||x|\leq r\big\}$ and let

$$I_n(r)=\int_{B_n(r)} f(x_1)f(x_2)\cdots f(x_n)\;dx_1\,dx_2\cdots dx_n.$$

Show that for all $r>0$ we have $\lim_{n\to\infty} I_n(r)=0$.

Solution: Let $r>0$. Let m be the maximum value of $f(x)$ for $|x|\leq r$. Let $V_n(r)$ be the n-volume of $B_n(r)$. Then for all $x=(x_1,x_2,\cdots,x_n)\in B_n(r)$, we have $|x_i|\leq r$ for all indices i so that $f(x_i)\leq m$ for all i and so

$$I_n(r)=\int_{B_n(r)} f(x_1)f(x_2)\cdots f(x_n)\;dx_1\,dx_2\,\cdots\,dx_n \leq \int_{B_n(r)} m^n\;dx_1\,dx_2\,\cdots\,dx_n = m^nV_n(r).$$

Since $r\,B_n(1)=B_n(r)$ we have $V_n(r)=r^nV_n(1)$. Thus $I_n(r)\leq m^nV_n(r)=(mr)^nV_n(1)$. Note that

$$\begin{aligned}V_{n+1}(1)&=\int_{B_{n+1}(1)} dx_1\,dx_2\,\cdots,dx_{n+1}=\int_{t=-1}^{1}\int_{B_n(\sqrt{1-t^2})} dx_1\,dx_2\,\cdots\,dx_n\\&=\int_{t=-1}^{1} V_n\big(\sqrt{1-t^2}\big)\,dt=\int_{t=-1}^{1}(1-t^2)^{n/2}V_n(1)\,dt=2\,V_n(1)\int_0^1(1-t^2)^{n/2}\,dt.\end{aligned}$$

For all ϵ with $0<\epsilon\leq 1$ we have

$$\begin{aligned}\frac{(mr)^{n+1}V_{n+1}(1)}{(mr)^nV_n(1)}&=2mr\int_0^1(1-t^2)^{n/2}\,dt=2mr\Big(\int_0^\epsilon(1-t^2)^{n/2}dt+\int_\epsilon^1(1-t^2)^{n/2}dt\Big)\\&\leq 2mr\Big(\int_0^\epsilon 1\,dt+\int_\epsilon^1(1-\epsilon^2)^{n/2}dt\Big)=2mr\Big(\epsilon+(1-\epsilon)(1-\epsilon^2)^{n/2}\Big)\\&\longrightarrow 2mr\epsilon \text{ as } n\to\infty.\end{aligned}$$

Since ϵ can be arbitrarily small, it follows that $\lim_{n\to\infty}\frac{(mr)^{n+1}V_{n+1}(1)}{(mr)^nV_n(1)}=0$. By the Ratio Test, $\sum_{n=1}^{\infty}(mr)^nV_n(1)$ converges, and so $\lim_{n\to\infty}(mr)^nV_n(1)=0$. Since $0\leq I_n(r)\leq (mr)^nV_n(1)$, it follows that $\lim_{n\to\infty} I_n(r)=0$ by the Squeeze Theorem.

We remark that we did not use the assumption that $\int_{-\infty}^{\infty} f(x)\,dx=1$.