SPECIAL K
Saturday November 5, 2016
10:00 am - 1:00 pm

1: Let S be the circle of radius 1 centred at O, let T be the circle of radius 1 centred at Q and suppose S and T are tangent at P. A ray from O intersects S at the point A then intersects T at the points B and C. Suppose the distance from A to B is equal to the distance from B to C. Find the area of triangle APB.

2: Let m be a positive integer. Let $a_1 = m$ and let $a_{n+1} = \left\lfloor \sqrt{n\, a_n} \right\rfloor$ for $n \geq 1$. Show that there exists a positive integer N such that for all $n \geq N$ we have $a_n = n - 3$.

3: For positive integers n and k, let $\sigma(n, k)$ be the sum of all the divisors d of n with $\frac{n}{k} \leq d \leq k$.

Find $S_k = \displaystyle\sum_{n=1}^{k^2} \sigma(n, k)$.

4: A game begins with a pile of n coins. Players A and B take turns with A going first. At each turn a player removes a nonzero perfect square number of coins from the pile. The player who removes the last coin wins. Show that there are infinitely many values of n for which player B has a winning strategy.

5: Let $f : [0, 1] \to \mathbf{R}$ with $f(0) > 0$ and $f(1) < 0$. Suppose that there exists a continuous function $g : [0, 1] \to \mathbf{R}$ such that $f + g$ is increasing. Show that there exists $c \in (0, 1)$ such that $f(c) = 0$.

6: Let m be a positive integer. Let S be a set of m-element subsets of $\mathbf{Z}$. Suppose that for all $A, B \in S$ we have $A \cap B \neq \emptyset$. Show that there exists a finite set $F \subseteq \mathbf{Z}$ such that for all $A, B \in S$ we have $A \cap B \cap F \neq \emptyset$.

BIG E
Saturday November 5, 2016
10:00 am - 1:00 pm

1: Define $f : \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x,y) = \big(4x - 3y + 1\,,\, 2x - y + 1\big)$. Find $f^n(2,1)$ where n is a positive integer and f^n is defined recursively by $f^1 = f$ and $f^k = f \circ f^{k-1}$.

2: Let p_k denote the k^{th} prime number. Show that $\sum\limits_{n=2}^{\infty} \sum\limits_{k=1}^{\infty} \dfrac{1}{{p_k}^n} < \frac{3}{2} - \ln 2$.

3: A game begins with a pile of n coins. Players A and B take turns with A going first. At each turn a player removes a nonzero perfect square number of coins from the pile. The player who removes the last coin wins. Show that there are infinitely many values of n for which player B has a winning strategy.

4: Let n be a positive integer. Find the largest integer m such that there exist m subsets $S_1, S_2, \cdots, S_m \subseteq \{1, 2, 3, \cdots, n\}$ with the property that each set S_k has an odd number of elements and each set $S_k \cap S_l$ with $k \neq l$ has an even number of elements.

5: Let m be a positive integer. Let S be a set of m-element subsets of $\mathbf{Z}$. Suppose that for all $A, B \in S$ we have $A \cap B \neq \emptyset$. Show that there exists a finite set $F \subseteq \mathbf{Z}$ such that for all $A, B \in S$ we have $A \cap B \cap F \neq \emptyset$.

6: Let $f : \mathbf{R} \to \mathbf{R}$ be continuous with $f(x) \geq 0$ for all $x \in \mathbf{R}$ and suppose $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $r > 0$ and $n \in \mathbf{Z}^+$, let $B_n(r) = \big\{x \in \mathbf{R}^n \big| |x| \leq r\big\}$ and let

$$I_n(r) = \int_{B_n(r)} f(x_1) f(x_2) \cdots f(x_n)\; dx_1\, dx_2 \cdots dx_n.$$

Show that for all $r > 0$ we have $\lim\limits_{n \to \infty} I_n(r) = 0$.