Solutions to the Special K Problems, 2012

1: Let $f(x) = x^4 + 2x^3$. Find the equation of a line which is tangent to the curve $y = f(x)$ at two distinct points.

Solution: We have $f'(x) = 4x^3 + 6x^2$. The tangent line to $y = f(x)$ at $x = a$ is given by $y = l(x)$ where

$$l(x) = f(x) + f'(a)(x-a) = (a^4 + 2a^3) + (4a^3 + 6a^2)(x-a) = (4a^3 + 6a^2)x - (3a^4 + 4a^3)\,.$$

Note that the function $g(x) = f(x) - l(x)$ has a double root at $x = a$, indeed

$$\begin{aligned} g(x) = f(x) - l(x) &= x^4 + 2x^3 - (4a^3 + 6a^2)x + (3a^4 + 4a^3) \\ &= (x-a)\big(x^3 + (a+2)x^2 + (a^2+2a)x - (3a^3 + 4a^2)\big) \\ &= (x-a)^2\big(x^2 + (2a+2)x + (3a^2 + 4a)\big)\,. \end{aligned}$$

In order for $y = l(x)$ to be tangent to the curve $y = f(x)$ at another point $\big(b, f(b)\big)$, we need $g(x)$ to have another double root at $x = b$. Since $g(x)$ is monic, it must be of the form $g(x) = (x-a)^2(x-b)^2$, so we must have

$$x^2 + (2a+2)x + (3a^2 + 4a) = (x-b)^2 = x^2 - 2b + b^2\,,$$

and so $b = -(a+1)$ (1) and $b^2 = 3a^2 + 4a$ (2). Put $b = -(a+1)$ into equation (2) to get $a^2+2a+1 = 3a^2+4a$, that is $2a^2+2a-1 = 0$. Thus $a = \frac{-2\pm\sqrt{4+8}}{4} = \frac{-1\pm\sqrt{3}}{2}$ and $b = -(a+1)$. When $a = \frac{-1+\sqrt{3}}{2}$ we have $b = \frac{-1-\sqrt{3}}{2}$, and vice versa. Taking $a = \frac{-1+\sqrt{3}}{2}$, we have $a^2 = \frac{2-\sqrt{3}}{2}$, $a^3 = \frac{-5+3\sqrt{3}}{4}$ and $a^4 = \frac{7-4\sqrt{3}}{4}$ and so the equation of the required tangent line is

$$y = l(x) = (4a^3 + 6a^2)x - (3a^4 + 4a^3) = \big((-5+3\sqrt{3}) + (6-3\sqrt{3})\big)x - \big(\tfrac{21-12\sqrt{3}}{4} + \tfrac{-20+12\sqrt{3}}{4}\big) = x - \tfrac{1}{4}\,.$$

2: Find the area of the region $R = \big\{(x,y) \in \mathbf{R}^2 \big|\, (x^2+y^2)^2 \le 4x^2 \text{ and } x(x^2+y^2) \le 2\sqrt{3}\,xy\,\big\}$.

Solution: When $x > 0$ we have $(x^2+y^2)^2 \le 4x^2 \iff x^2 + y^2 \le 2x \iff (x-1)^2 + y^2 \le 1$ and we have $x(x^2+y^2) \le 2\sqrt{3}\,xy \iff x^2+y^2 \le \sqrt{3}\,y \iff x^2 + (y-\sqrt{3})^2 \le 3$, and so the part of the region R which lies to the right of the y-axis is the region A which lies inside both the circle centered at $(1,0)$ of radius 1 and the circle centered at $(0,\sqrt{3})$ of radius $\sqrt{3}$. When $x < 0$, on the other hand, we have $(x^2+y^2)^2 \le 4x^2 \iff x^2+y^2 \le -2x \iff (x+1)^2+y^2 \le 1$ and $x(x^2+y^2) \le 2\sqrt{3}\,xy \iff x^2+y^2 \ge 2\sqrt{3}\,y \iff x^2 + (y-\sqrt{3})^2 \ge 3$ and so the part of the region R which lies to the left of the y-axis is the region B which lies inside the circle centred at $(-1,0)$ of radius 1 and outside the circle centered at $(0,\sqrt{3})$ of radius $\sqrt{3}$. The area of R is the sum of the areas of A and B which, by symmetry, is equal to the areal of a unit circle, namely π.

3: Let x_n be the number of $2\times n$ matrices with entries in $\{0,1\}$ which do not contain the 2×2 block $\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

Find $\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n}$.

Solution: Let a_n, b_n, c_n and d_n be the number of allowable $2\times n$ matrices which end with the column $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$. Note that $a_1=b_1=c_1=d_1=1$. Each of the three columns $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ can be appended to any allowable $2\times n$ matrix to get an allowable $2\times(n+1)$ matrix, so we have

$$a_{n+1}=c_{n+1}=d_{n+1}=a_n+b_n+c_n+d_n.$$

It follows that $a_n=c_n=d_n$ for all $n\geq 1$, and we can write the above recursion formula as

$$a_{n+1}=3a_n+b_n\,.$$

The column $\begin{pmatrix}0\\1\end{pmatrix}$ can be appended to any allowable $2\times n$ matrix which does not end with $\begin{pmatrix}1\\0\end{pmatrix}$, so we have

$$b_{n+1}=a_n+b_n+d_n=2a_n+b_n\,.$$

From the formula $a_{n+1}=3a_n+b_n$ we get $b_n=a_{n+1}-3a_n$ $\big($hence also $b_{n+1}=a_{n+2}-3a_{n+1}\big)$. Put this into the formula $b_{n+1}=2a_n+b_n$ to get $a_{n+2}-3a_{n+1}=2a_n+a_{n+1}-3a_n$ which we can also write as

$$a_{n+2}=4a_{n+1}-a_n\,.$$

Note that $x_n=a_n+b_n+c_n+d_n=3a_n+b_n=a_{n+1}$, so that $x_1=4$, $x_2=a_3=15$ and for $n\geq 2$ we have

$$x_{n+1}=4x_n-x_{n-1}\,.$$

Dividing by x_n gives

$$\frac{x_{n+1}}{x_n}=4-\frac{x_{n-1}}{x_n}\,.$$

The above formula shows that $\left\{\frac{x_{n+1}}{x_n}\right\}$ is decreasing, and we have $x_{n+1}=a_{n+2}=3a_{n+1}+b_{n+1}\geq 3a_{n+1}=3\,x_n$ so that $\frac{x_{n+1}}{x_n}\geq 3$, and so the sequence $\left\{\frac{x_{n+1}}{x_n}\right\}$ must converge with $\lim\limits_{n\to\infty}\dfrac{x_{n+1}}{x_n}\geq 3$. Let $L=\lim\limits_{x\to\infty}\dfrac{x_{n+1}}{x_n}$. By taking the limit on both sides of the formula $\dfrac{x_{n+1}}{x_n}=4-\dfrac{x_{n-1}}{x_n}$ we obtain $L=4-\frac{1}{L}$, that is $L^2-4L+1=0$, and so $L=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{2}$. Since $L\geq 3$ we must have $L=2+\sqrt{2}$.

4: Let $k\geq 3$ be an integer. Let $n=\frac{k(k+1)}{2}$. Let $S\subseteq\mathbf{Z}_n$ with $|S|=k$. Show that $S+S\neq\mathbf{Z}_n$.

Solution: Say $S=\big\{a_1,a_2,\cdots,a_k\big\}$. Then each element of $S+S$ is of the form a_j+a_k for some 1 or 2-element subset $\{a_j,a_k\}\subset S$ (where we allow the possibility that $a_j=a_k$). There are $\frac{k(k+1)}{2}$ such subsets, and so to show that $S+S\neq\mathbf{Z}_n$ it suffices to find two distinct sets $\{a_i,a_l\}\neq\{a_j,a_k\}$ with $a_i+a_l=a_j+a_k$.

There are $k(k-1)$ ordered pairs (a_i,a_j) with $a_i\neq a_j$. For such pairs, there are $n-1$ possible values for the difference a_i-a_j in $\mathbf{Z}_n$ (since the difference cannot be zero). For $k\geq 3$ we have

$$k(k-1)=\tfrac{k(k+1)}{2}+\tfrac{k(k-3)}{2}\geq\tfrac{k(k+1)}{2}=n>n-1$$

so by the Pigeonhole principle, we can choose two order pairs $(a_i,a_j)\neq(a_k,a_l)$ with $a_i\neq a_j$ and $a_k\neq a_l$ such that $a_i-a_j=a_k-a_l$. Note that $a_i+a_l=a_j+a_k$ and note that $\{a_i,a_l\}\neq\{a_j,a_k\}$ $\big($indeed, if we had $\{a_i,a_l\}=\{a_j,a_k\}$ then since $a_i\neq a_j$ we would need $a_i=a_k$, and since $a_l\neq a_k$ we would need $a_l=a_j$, but then we would have $(a_i,a_j)=(a_k,a_l)\big)$.

5: Let $f:\mathbf{R}\to\mathbf{R}$. Suppose that $\lim\limits_{x\to 0} f(x)=f(0)=0$ and $\lim\limits_{x\to 0}\dfrac{f(2x)-f(x)}{x}=0$. Show that f is differentiable at 0 with $f'(0)=0$.

Solution: Let $\epsilon>0$. Choose $\delta>0$ so that $0<|x|<\delta\Longrightarrow\left|\dfrac{f(2x)-f(x)}{x}\right|<\dfrac{\epsilon}{2}$. Let $x\in\mathbf{R}$ with $0<|x|<\delta$. Note that for $k\in\mathbf{Z}^+$ we have $0<\left|\frac{x}{2^k}\right|<\delta$ and so $\left|\dfrac{f\left(\frac{x}{2^{k-1}}\right)-f\left(\frac{x}{2^k}\right)}{\frac{x}{2^k}}\right|<\dfrac{\epsilon}{2}$, hence $\left|\dfrac{f\left(\frac{x}{2^{k-1}}\right)-f\left(\frac{x}{2^k}\right)}{x}\right|<\dfrac{\epsilon}{2^{k+1}}$. Thus for all $n\in\mathbf{Z}^+$ we have

$$\begin{aligned}\left|\frac{f(x)-f(0)}{x-0}\right|=\left|\frac{f(x)}{x}\right|&=\left|\frac{f(x)-f\left(\frac{x}{2}\right)}{x}+\frac{f\left(\frac{x}{2}\right)-f\left(\frac{x}{4}\right)}{x}+\cdots+\frac{f\left(\frac{x}{2^{n-1}}\right)-f\left(\frac{x}{2^n}\right)}{x}+\frac{f\left(\frac{x}{2^n}\right)}{x}\right|\\ &\leq\left|\frac{f(x)-f\left(\frac{x}{2}\right)}{x}\right|+\left|\frac{f\left(\frac{x}{2}\right)-f\left(\frac{x}{4}\right)}{x}\right|+\cdots+\left|\frac{f\left(\frac{x}{2^{n-1}}\right)-f\left(\frac{x}{2^n}\right)}{x}\right|+\left|\frac{f\left(\frac{x}{2^n}\right)}{x}\right|\\ &<\frac{\epsilon}{4}+\frac{\epsilon}{8}+\cdots+\frac{\epsilon}{2^{n+1}}+\left|\frac{f\left(\frac{x}{2^n}\right)}{x}\right|<\frac{\epsilon}{2}+\frac{\left|f\left(\frac{x}{2^n}\right)\right|}{|x|}\,.\end{aligned}$$

In particular, choosing n large enough so that $\left|f\left(\frac{x}{2^n}\right)\right|<\frac{\epsilon|x|}{2}$ $\big($which we can do since $\lim\limits_{x\to 0} f(x)=0\big)$ we have

$$\left|\frac{f(x)-f(0)}{x-0}\right|<\epsilon\,.$$

6: Let $\mathbf{Z}^+$ be the set of positive integers. Show that there exists a bijection $f:\mathbf{Z}^+\to\mathbf{Z}^+$ with the property that $\prod\limits_{k=1}^{n} f(k)$ is an n^{th} power for every $n\in\mathbf{Z}^+$.

Solution: We construct such a bijection. We define $f(1)=1$. Having defined $f(1),f(2),\cdots,f(2n-1)$, we define $f(2n)$ and $f(2n+1)$ as follows. First we define $f(2n+1)$ to be the smallest positive integer with $f(2n+1)\notin\big\{f(1),f(2),\cdots,f(2n-1)\big\}$, and then we define

$$f(2n)=\big(f(1)f(2)\cdots f(2n-1)\big)^{(2n)(2n+1)-1}f(2n+1)^{2n}\,.$$

Solutions to the Big E Problems, 2012

1: Find the volume of the solid $S = \left\{ (x,y,z) \in \mathbf{R}^3 \middle| \, (x^2+y^2+z^2)^2 \leq 4x^2 \text{ and } x(x^2+y^2) \leq xz^2 \right\}$.

Solution: When $x > 0$ we have $(x^2+y^2+z^2)^2 \leq 4x^2 \iff x^2+y^2+z^2 \leq 2x \iff (x-1)^2+y^2+z^2 \leq 1$ and we have $x(x^2+y^2) \leq xz^2 \iff x^2+y^2 \leq z^2$, and so the part of the solid S which lies to the right of the yz-plane is the region A which lies inside both the sphere centered at $(1,0,0)$ of radius 1 and the double cone $x^2+y^2=z^2$. When $x<0$, on the other hand, we have $(x^2+y^2+z^2)^2 \leq 4x^2 \iff x^2+y^2+z^2 \leq -2x$ $\iff (x+1)^2+y^2+z^2 \leq 1$ and $x(x^2+y^2) \leq xz^2 \iff x^2+y^2 \geq z^2$ and so the part of the solid S which lies to the left of the yz-plane is the region B which lies inside the sphere centred at $(-1,0,0)$ of radius 1 and outside the double cone $x^2+y^2=z^2$. The volume of S is the sum of the volumes of A and B which, by symmetry, is equal to the volume of a unit sphere, namely $\frac{4\pi}{3}$.

2: Find the number of $3 \times n$ matrices with entries in $\{0,1\}$ which do not contain the 2×2 block $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution: For $k = 0,1,2,\cdots,7$, let $a_{k,n}$ be the number of allowable $3 \times n$ matrices ending with the column which corresponds to the binary representation of k. Note that $a_{k,1} = 1$ for all k. Since each of the columns $(0,0,0)^T$,$(1,0,0)^T$, $(1,1,0)^T$, $(1,1,1)^T$ can be appended to any allowable $3 \times n$ matrix to obtain an allowable $3 \times (n+1)$ matrix, we have

$$a_{0,n+1} = a_{4,n+1} = a_{6,n+1} = a_{7,n+1} = a_{0,n} + a_{1.n} + a_{2,n} + \cdots + a_{7,n} \,.$$

Since each of the columns $(0,0,1)^T$,$(1,0,1)^T$ can be appended to any allowable $3 \times n$ matrix with any final column other than $(0,1,0)^T$ or $(1,1,0)^T$ we have

$$a_{1,n+1} = a_{5,n+1} = a_{0,n} + a_{1,n} + a_{3,n} + a_{4,n} + a_{5,n} + a_{7,n} \,.$$

Since each of the columns $(0,1,0)^T$,$(0,1,1)^T$ can be appended to any allowable $3 \times n$ matrix with any final column other than $(1,0,0)^T$,$(1,0,1)^T$ we have

$$a_{2,n+1} = a_{3,n+1} = a_{0,n} + a_{1,n} + a_{2,n} + a_{3,n} + a_{6,n} + a_{7n} \,.$$

We see that $a_{0,n} = a_{4,n} = a_{6,n} = a_{7,n}$ for all n, and $a_{1,n} = a_{5,n}$ for all n, and $a_{2,n} = a_{3,n}$ for all n. Say $a_n = a_{0,n}$, $b_n = a_{1,n}$ and $c_n = a_{2,n}$. Then we have $a_1 = b_1 = c_1$ and the above recursion formulas simplify to

$$\begin{aligned} a_{n+1} &= 4a_n + 2b_n + 2c_n \\ b_{n+1} &= 3a_n + 2b_n + c_n \\ c_{n+1} &= 3a_n + b_n + 2c_n \,. \end{aligned}$$

By the symmetry between b and c in these equations we see that $b_n = c_n$ for all n, so the formulas further simplify to

$$\begin{aligned} a_{n+1} &= 4a_n + 4b_n \\ b_{n+1} &= 3a_n + 3b_n = \tfrac{3}{4}\, a_{n+1} \,. \end{aligned}$$

Thus we have $a_1 = 1$, $b_1 = 1$, $a_2 = 8$, $b_2 = 7$, and for $n \geq 1$ we have $b_n = \frac{3}{4}\, a_n$ so that

$$a_{n+1} = 4a_n + 4b_n = 4a_n + 3a_n = 7a_n \,.$$

Thus for $n \geq 2$ we have $a_n = 8 \cdot 7^{n-2}$ and $b_n = 7 \cdot 7^{n-2}$. For $n \geq 1$, the total number of allowable $3 \times n$ matrixes is equal to

$$4a_n + 4b_n = a_{n+1} = 8 \cdot 7^{n-1} \,.$$

3: Let $k \geq 3$ be an integer. Let $n = \frac{k(k+1)}{2}$. Let $S \subseteq \mathbf{Z}_n$ with $|S| = k$. Show that $S + S \neq \mathbf{Z}_n$.

Solution: Say $S = \{a_1, a_2, \cdots, a_k\}$. Then each element of $S+S$ is of the form $a_j + a_k$ for some 1 or 2-element subset $\{a_j, a_k\} \subset S$ (where we allow the possibility that $a_j = a_k$). There are $\frac{k(k+1)}{2}$ such subsets, and so to show that $S + S \neq \mathbf{Z}_n$ it suffices to find two distinct sets $\{a_i, a_l\} \neq \{a_j, a_k\}$ with $a_i + a_l = a_j + a_k$.

There are $k(k-1)$ ordered pairs (a_i, a_j) with $a_i \neq a_j$. For such pairs, there are $n - 1$ possible values for the difference $a_i - a_j$ in $\mathbf{Z}_n$ (since the difference cannot be zero). For $k \geq 3$ we have

$$k(k-1) = \tfrac{k(k+1)}{2} + \tfrac{k(k-3)}{2} \geq \tfrac{k(k+1)}{2} = n > n - 1$$

so by the Pigeonhole principle, we can choose two order pairs $(a_i, a_j) \neq (a_k, a_l)$ with $a_i \neq a_j$ and $a_k \neq a_l$ such that $a_i - a_j = a_k - a_l$. Note that $a_i + a_l = a_j + a_k$ and note that $\{a_i, a_l\} \neq \{a_j, a_k\}$ (indeed, if we had $\{a_i, a_l\} = \{a_j, a_k\}$ then since $a_i \neq a_j$ we would need $a_i = a_k$, and since $a_l \neq a_k$ we would need $a_l = a_j$, but then we would have $(a_i, a_j) = (a_k, a_l)$).

4: Let $f : \mathbf{R}^2 \to \mathbf{R}$. Suppose that f is continuous and that $\displaystyle\int_0^1 f(a + tu)\, dt = 0$ for every point $a \in \mathbf{R}^2$ and every vector $u \in \mathbf{R}^2$ with $|u| = 1$. Show that f is constant.

Solution: Let $a, u \in \mathbf{R}^2$ and with $|u| = 1$. For $x \in \mathbf{R}$, the substitution $t = s + x$ gives

$$\int_x^{1+x} f(a + tu)\, du = \int_0^1 f\big(a + xu + su\big)\, ds = 0$$

and so we have

$$\int_0^x f(a + tu)\, dt - \int_1^{1+x} f(a + tu)\, dt = \int_0^1 f(a + tu)\, dt - \int_x^{1+x} f(a + tu)\, dt = 0\,.$$

Differentiate both sides with respect to x using the FTC to get $f(a+xu) - f(a+xu+u) = 0$. In particular, taking $x = 0$, we obtain

$$f(a) = f(a + u)\,.$$

To show that f is constant, we shall show that $f(a) = f(0)$ for all $a \in \mathbf{R}^2$. Given $a \in \mathbf{R}^2$, let $k = \big\lfloor |a| \big\rfloor$, let $u = \frac{a}{|a|}$ and let $b = a - ku$. Then we have $|b| < 1$ and

$$f(a) = f(a - u) = f(a - 2u) = \cdots = f(a - ku) = f(b)\,.$$

Let v and w be the two points of intersection of the unit circle with the perpendicular bisector of the line segment from 0 to b so that $|v| = |w| = 1$ and $v + w = b$. Then $f(0) = f(v) = f(v + w) = f(b) = f(a)$.

5: Let $\mathbf{Z}^+$ be the set of positive integers. Show that there exists a bijection $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ with the property that $\prod_{k=1}^{n} f(k)$ is an n^{th} power for every $n \in \mathbf{Z}^+$.

Solution: We construct such a bijection. We define $f(1) = 1$. Having defined $f(1), f(2), \cdots, f(2n-1)$, we define $f(2n)$ and $f(2n+1)$ as follows. First we define $f(2n+1)$ to be the smallest positive integer with $f(2n+1) \notin \{f(1), f(2), \cdots, f(2n-1)\}$, and then we define

$$f(2n) = \big(f(1)f(2)\cdots f(2n-1)\big)^{(2n)(2n+1)-1} f(2n+1)^{2n}\,.$$

6: Let A be an $n \times n$ matrix. Let u be an eigenvector of A for the eigenvalue 1. Suppose that all of the entries of A and all of the entries of u are positive. Show that the eigenspace for the eigenvalue 1 is 1-dimensional.

Solution: Let v be any eigenvector for the eigenvalue 1. We must show that $u = cv$ for some $0 \neq c \in \mathbf{R}$. Suppose that v has at least one positive entry (otherwise replace v by $-v$). Choose k with $v_k > 0$ to minimize $\frac{u_k}{v_k}$ (so we have $\frac{u_k}{v_k} \leq \frac{u_i}{v_i}$ whenever $v_i > 0$). We claim that $u = \frac{u_k}{v_k} v$. Consider the vector $w = u - \frac{u_k}{v_k} v$. The i^{th} entry of w is $w_i = u_i - \frac{u_k}{v_k} v_i$. If $v_i \leq 0$ then we have $w_i \geq u_i > 0$, and if $v_i > 0$ then we have $w_i = \big(\frac{u_i}{v_i} - \frac{u_k}{v_k}\big) v_i \geq 0$, so we have $w_i \geq 0$ for all i. Also note that

$$Aw = A\big(u - \tfrac{u_k}{v_k}\big) = Au - \tfrac{u_k}{v_k} Av = u - \tfrac{u_k}{v_k} v = w\,.$$

Suppose, for a contradiction, that $w \neq 0$. Then each entry $w_i \geq 0$ and some entry $w_l > 0$. Since every entry of A is positive, it follows that every entry of Aw is positive, indeed the i^{th} entry of Aw is

$$\big(Aw\big)_i = \sum_{j=1}^{n} A_{i,j} w_j \geq A_{i,l} w_l > 0\,.$$

Since $w = Aw$, every entry of w is positive. But this is not possible since $w_k = u_k - \frac{u_k}{v_k} v_k = 0$. Thus $w = 0$ and so we have $u = \frac{u_k}{v_k} v$, as claimed.