Solutions to the Special K Problems, 2011

1: Find the number of sequences $a_1, a_2, \cdots, a_6$ with each $a_i \in \{1,2,3,4\}$ such that

$$a_1 < a_2 \ , \ a_2 > a_3, \ a_3 < a_4 \ , \ a_4 > a_5 \ , \ a_5 < a_6 \ \text{ and } \ a_6 > a_1 \, .$$

Solution: For $1 \leq l \leq 6$, let us call a sequence $(a_1, a_2, \cdots, a_l)$ *allowable* when each $a_i \in \{1,2,3,4\}$ and $a_1 < a_2\,,\, a_2 > a_3\,,\, a_3 < a_4\,, \cdots$. For $1 \leq l \leq 6$ and $j,k \in \{1,2,3,4\}$ let $c^k_{l,j}$ be the number of allowable sequences $(a_1, \cdots, a_l)$ with $a_1 = k$ and $a_l = j$. Then clearly $c^k_{1,k} = 1$ and $c^k_{1,j} = 0$ when $j \neq k$, and then we can obtain all the values $c^k_{l,j}$ recursively as follows. Fix k and suppose we have found $c^k_{l-1,i}$ for all i. If l is even then we need $a_{l-1} < a_l$ and we can append $a_l = j$ to any allowable sequence $(a_1, \cdots, a_{l-1})$ with $a_{l-1} = i < j$, and so we have $c^k_{l,j} = \sum\limits_{i<j} c^k_{l-1,i}$ (in the case that $j = 1$ we obtain the empty sum which we take to be zero). Similarly, if l is odd we have $c^k_{l,j} = \sum\limits_{i>j} c_{l-1,i}$ (in the case that $j = 4$ we take the sum to be zero). We use this recursion to find $c^k_{l,j}$ for all j,k,l (except $k = 4$, which we do not need).

$c^1_{l,j}$

4	0	1	0	6	0	31
3	0	1	1	5	6	25
2	0	1	2	3	11	14
1	1	0	3	0	14	0
j/l	1	2	3	4	5	6

$c^2_{l,j}$

4	0	1	0	5	0	25
3	0	1	1	4	5	20
2	1	0	2	2	9	11
1	0	0	2	0	11	0
j/l	1	2	3	4	5	6

$c^3_{l,j}$

4	0	1	0	3	0	14
3	1	0	1	2	3	11
2	0	0	1	1	5	6
1	0	0	1	0	6	0
j/l	1	2	3	4	5	6

The total number of allowable sequences $(a_1, a_2, \cdots, a_6)$ with $a_6 > a_1$ is equal to

$$c^1_{6,2} + c^1_{6,3} + c^1_{6,4} + c^2_{6,3} + 2^2_{6,4} + c^3_{6,4} = 14 + 25 + 31 + 20 + 25 + 14 = 129.$$

2: Find the number of solutions to the congruence $x^2 \equiv 40\,x \mod 10^6$.

Solution: First we solve $x^2 \equiv 40\,x \mod 2^6$. Since $40\,x$ is even, to get $x^2 \equiv 40\,x \mod 2^6$ we must have x even, say $x = 2y$. Then

$$x^2 \equiv 40\,x \mod 2^6 \iff 4y^2 \equiv 80\,y \mod 2^6 \iff y^2 \equiv 20\,y \mod 2^4.$$

Since $20\,y$ is even we must have y even, say $y = 2z$. Then

$$y^2 \equiv 20y \mod 2^4 \iff 4z^2 \equiv 40z \mod 2^4 \iff z^2 \equiv 10z \mod 2^2 \iff z^2 \equiv 2\,z \mod 4 \iff z = 0 \mod 2$$
$$\iff y = 2z = 0 \mod 4 \iff x = 2y \equiv 0 \mod 8.$$

Thus there are 8 solutions to $x^2 \equiv 40\,x \mod 2^6$, namely $x = 0, 8, 16, 24, 32, 40, 48, 56 \mod 64$.

Next we solve $x^2 \equiv 40\,x \mod 5^6$. Since $40\,x$ is a multiple of 5, to get $x^2 \equiv 40\,x \bmod 5^6$ it must be that x is a multiple of 5, say $x = 5y$. Then

$$x^2 = 40\,x \mod 5^6 \iff 25\,y^2 \equiv 200\,y \mod 5^6 \iff y^2 \equiv 8\,y \mod 5^4.$$

If y is not a multiple of 5, then y is invertible modulo 5, so we can divide by y to get

$$y^2 \equiv 8\,y \mod 5^4 \iff y \equiv 8 \mod 5^4 \iff x = 5y \equiv 40 \mod 5^5.$$

Suppose that y is a multiple of 5, say $y = 5z$. Then

$$y^2 \equiv 8\,y \mod 5^4 \iff 25\,z^2 \equiv 40\,z \mod 5^4 \iff 5\,z^2 \equiv 8\,z \mod 5^3.$$

Since $5\,z^2$ is a multiple of 5 it must be that z is a multiple of 5, say $z = 5w$. Then

$$5\,z^2 \equiv 8\,z \mod 5^3 \iff 0 \equiv 40\,w \mod 5^3 \iff 8\,w \equiv 0 \mod 5^2 \iff w \equiv 0 \mod 5^2$$
$$\iff z = 5w \equiv 0 \mod 5^3 \iff y = 5z \equiv 0 \bmod 5^4 \iff x = 5y \equiv 0 \mod 5^5.$$

Thus there are 10 solutions to $x^2 \equiv 40\,x \mod 5^6$, namely $x \equiv 0, 40 \mod 5^5$.

Since the congruence $x^2 = 40\,x \mod 10^6$ has 8 solutions modulo 2^6 and 10 solutions modulo 5^6, by the Chinese Remainder Theorem, it has $8 \cdot 10 = 80$ solutions modulo 10^6.

3: Four spheres of radius 1 are inscribed in a regular tetrahedron so that each sphere is tangent to the other 3 spheres and to 3 of the faces of the tetrahedron. Find the length of the edges of the tetrahedron.

Solution: Let S be the regular tetrahedron which contains the spheres. Let T be the smaller regular tetrahedron whose vertices are the centres of the four spheres. Let O be the common centre of S and T. The edges of T have length 2. The faces of S are parallel to, and 1 unit away from, the faces of T. Let r be the distance from the centre O to the faces of T (in other words, let r be the radius of the inscribed sphere of T). Then the faces of S lie at a distance of $1+r$ units from O, and so S is obtained by scaling T by a factor of $\frac{1+r}{r} = \frac{1}{r}+1$, and so the sides of S are of length $2\left(\frac{1}{r}+1\right)$. Thus it suffices to find the value of r.

The height of each triangular face (that is the length of each median of each face) of the tetrahedron T is $\sqrt{3}$. The centroid of the base triangle of T lies one third of the way along each median, so the height h (that is the length of the altitude) of T is given (by Pythagoras' Theorem) by $h^2 = (\sqrt{3})^2 - \left(\frac{1}{3}\sqrt{3}\right)^2 = 3 - \frac{1}{3} = \frac{8}{3}$, that is $h = \frac{2\sqrt{2}}{\sqrt{3}}$. The centroid of the tetrahedron lies one quarter of the way along the altitude and so we have $r = \frac{1}{4}h = \frac{1}{\sqrt{6}}$. Thus the sides of S are of length $2\left(\frac{1}{r}+1\right) = 2(\sqrt{6}+1)$.

4: Let S be a set of 4 distinct real numbers. Show that there exist $a, b \in S$ such that

$$0 < \frac{a-b}{1+ab} \le 1\,.$$

Solution: Let $S = \{x_1, x_2, x_3, x_4\}$ with $x_1 < x_2 < x_3 < x_4$. Let $\theta_i = \tan^{-1}(x_i)$ for each i so we have $-\frac{\pi}{2} < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \frac{\pi}{2}$. Let $\phi_1 = \theta_2 - \theta_1$, $\phi_2 = \theta_3 - \theta_2$, $\phi_3 = \theta_4 - \theta_3$ and $\phi_4 = \pi + \theta_1 - \theta_4$. Then we have $0 < \phi_i$ for all i and we have $\phi_1 + \phi_2 + \phi_3 + \phi_4 = \pi$. It follows that $\phi_i \le \frac{\pi}{4}$ for some i. Choose k so that $\phi_k \le \frac{\pi}{4}$. Since $0 < \phi_k \le \frac{\pi}{4}$ we have $0 < \tan\phi_k \le 1$. Finally note that if $k \in \{1,2,3\}$ then we have

$$\tan\phi_k = \tan(\theta_{k+1} - \theta_k) = \frac{\tan\theta_{k+1} - \tan\theta_k}{1 + \tan\theta_{k+1}\tan\theta_k} = \frac{x_{k+1} - x_k}{1 + x_{k+1}x_k}$$

and if $k = 4$ then we have

$$\tan\phi_k = \tan(\pi + \theta_1 - \theta_4) = \tan(\theta_1 - \theta_4) = \frac{\tan\theta_1 - \tan\theta_4}{1 + \tan\theta_1\tan\theta_4} = \frac{x_1 - x_4}{1 + x_1x_4}\,.$$

5: Let $f(z)$ be a polynomial with complex coefficients of degree $n \ge 1$. Show that there exist at least $n+1$ distinct complex numbers z with $f(z) \in \{0,1\}$.

Solution: For a finite set A, let $|A|$ denote the number of elements in A. For a polynomial $f(x)$ and for $a \in \mathbf{C}$, let $\text{mult}(f,a)$ denote the smallest $m \in \mathrm{N}$ such that $(x-a)^m$ is a factor of $f(x)$. (in particular $\text{mult}(f,a) = 0$ when a is not a root of f). For a polynomial $f(x) = \sum_{k=0}^{n} c_k x^k$, let $f'(x)$ denote the derivative of f, which is given by $f'(x) = \sum_{k=0}^{n} k\, c_k x^{k-1}$. As with real polynomials, we have the product rule $(fg)' = f'g + fg'$ and we have $\text{mult}(f,a) = \text{mult}(f',a) + 1$. By the Fundamental Theorem of Algebra, for complex polynomials (but not for real polynomials) we have $\sum_{a\in\mathbf{C}} \text{mult}(f,a) = \deg(f)$. Let $g(x) = f(x) - 1$. Note that $g'(x) = f'(x)$. Let $A = \{a \in \mathbf{C} \,|\, f(a) = 0\}$ and let $B = \{b \in \mathbf{C} \,|\, f(b) = 1\} = \{b \in \mathbf{C} \,|\, g(b) = 0\}$. Note that $A \cap B = \emptyset$. Then

$$\begin{aligned}
|A| &= \sum_{a\in A} 1 = \sum_{a\in A} \big(\text{mult}(f,a) - \text{mult}(f',a)\big) \\
|B| &= \sum_{b\in B} 1 = \sum_{b\in B} \big(\text{mult}(g,b) - \text{mult}(g',b)\big) \\
|A| + |B| &= \sum_{a\in A} \text{mult}(f,a) + \sum_{b\in B} \text{mult}(g,b) - \sum_{a\in A} \text{mult}(f',a) - \sum_{b\in B} \text{mult}(f',b) \\
&\ge \sum_{a\in A} \text{mult}(f,a) + \sum_{b\in B} \text{mult}(g,b) - \sum_{c\in\mathbf{C}} \text{mult}(f',c) \\
&= \deg(f) + \deg(g) - \deg(f') = n + n - (n-1) = n+1.
\end{aligned}$$

6: Let $f : [0,1] \to [0,1]$ be continuous. Let $a_0 = 0$ and for $n \geq 1$ let $a_n = f(a_{n-1})$. Suppose $\lim\limits_{n\to\infty}(a_n - a_{n-1}) = 0$. Show that the sequence $\{a_n\}$ converges.

Solution: Suppose, for a contradiction, that $\{a_n\}$ diverges. Let $a = \liminf a_n$ and let $b = \limsup a_n$. Since $\{a_n\}$ diverges, we have $a < b$. Since the sequence $\{a_n\}$ oscillates back and forth, at times approaching a and at times approaching b, it follows, we claim, that there is some term $a_k \in \left[a + \frac{b-a}{3}, b - \frac{b-a}{3}\right]$. Here is a formal proof of this claim. Choose n_0 so that $n > n_0 \implies |a_n - a_{n-1}| < \frac{b-a}{3}$ (we can do this since $|a_n - a_{n-1}| \to 0$). Choose $n_1 > n_0$ so that $a_{n_1} < a + \frac{b-a}{3}$ (we can do this since $\liminf a_n = a$). Choose $n_2 > n_1$ so that $a_{n_2} > b - \frac{b-a}{3}$ (we can do this since $\limsup a_n = b$). Choose k with $n_1 < k \leq n_2$ so that $a_{n_1}, a_{n_1+1}, \cdots, a_{k-1} < a + \frac{b-a}{3}$ and $a_k \geq a + \frac{b-a}{3}$. By our choice of k we have

$$a + \tfrac{b-a}{3} \leq a_k < a_{k-1} + \tfrac{b-a}{3} < a + \tfrac{b-a}{3} + \tfrac{b-a}{3} = b - \tfrac{b-a}{3},$$

proving the claim. Note that $f(a_k) \neq a_k$, since otherwise the sequence $\{a_n\}$ would become constant for $n \geq k$. Let $\epsilon = \frac{1}{3}|f(a_k) - a_k|$, and choose $\delta > 0$ with $\delta \leq \epsilon$ and $\delta \leq \frac{b-a}{3}$ $\big($so that $(a_k - \delta, a_k + \delta) \subseteq (a,b)\big)$ such that $|x - a_k| < \delta \implies \big|f(x) - f(a_k)\big| < \epsilon$ (we can do this since f is continuous). Note that for $|x - a_k| < \delta$ we have $|x - a_k| < \epsilon$ and $\big|f(x) - f(a_k)\big| < \epsilon$ and so

$$3\epsilon = \big|f(a_k) - a_k\big| \leq \big|f(a_k) - f(x)\big| + \big|f(x) - x\big| + |x - a_k| < 2\epsilon + |f(x) - x|$$

and hence $\big|f(x) - x\big| > \epsilon$. Choose l_0 so that $n > l_0 \implies |a_l - a_{l-1}| < \epsilon$. Since the sequence $\{a_n\}$ oscillates back and forth, at times approaching a and at times approaching b, it follows (as above) that we can choose $l > l_0$ with $a_l \in \big(a_k - \delta, a_k + \delta\big)$. This gives the desired contradiction because since $l > l_0$ we have $|f(a_l) - a_l| = |a_{l+1} - a_l| < \epsilon$ but since $|a_l - a_k| < \delta$ we have $|f(a_l) - a_l| > \epsilon$.

Solutions to the Big E Problems, 2010

1: Find $\displaystyle\int_0^e x\, e^{-\lfloor \ln x\rfloor}\, dx$. (For $y \in \mathbf{R}$, $\lfloor y\rfloor$ denotes the largest integer n with $n \le y$).

Solution: Note that for $0 < x \in \mathbf{R}$ and $k \in \mathbf{Z}$ we have

$$e^{-k} \le x < e^{-(k-1)} \iff -k \le \ln x < -(k-1) \iff \lfloor \ln x\rfloor = -k$$

and so

$$\begin{aligned}\int_0^e x\, e^{-\lfloor \ln x\rfloor}\, dx &= \sum_{k=0}^{\infty} \int_{e^{-k}}^{e^{-(k-1)}} x\, e^k\, dx = \sum_{k=0}^{\infty} e^k \left[\tfrac{1}{2}\, x^2\right]_{e^{-k}}^{e^{-(k-1)}} = \sum_{k=0}^{\infty} \tfrac{1}{2}\, e^k \left(e^{-2k+2} - e^{-2k}\right)\\ &= \tfrac{1}{2}\sum_{k=0}^{\infty}\left(e^{-(k-2)} - e^{-k}\right) = \tfrac{1}{2}\Big((e^2-1)+(e-e^{-1})+(1-e^{-2})+(e-e^{-3})+\cdots\Big)\\ &= \tfrac{1}{t}(e^2+e),\end{aligned}$$

since the sum telescopes (which means that almost all the terms cancel).

2: Four spheres of radius 1 are inscribed in a regular tetrahedron so that each sphere is tangent to the other 3 spheres and to 3 of the faces of the tetrahedron. Find the length of the edges of the tetrahedron.

Solution: Let S be the regular tetrahedron which contains the spheres. Let T be the smaller regular tetrahedron whose vertices are the centres of the four spheres. Let O be the common centre of S and T. The edges of T have length 2. The faces of S are parallel to, and 1 unit away from, the faces of T. Let r be the distance from the centre O to the faces of T (in other words, let r be the radius of the inscribed sphere of T). Then the faces of S lie at a distance of $1+r$ units from O, and so S is obtained by scaling T by a factor of $\frac{1+r}{r} = \frac{1}{r}+1$, and so the sides of S are of length $2\big(\frac{1}{r}+1\big)$. Thus it suffices to find the value of r.

The height of each triangular face (that is the length of each median of each face) of the tetrahedron T is $\sqrt{3}$. The centroid of the base triangle of T lies one third of the way along each median, so the height h (that is the length of the altitude) of T is given (by Pythagoras' Theorem) by $h^2 = (\sqrt{3})^2 - \big(\frac{1}{3}\sqrt{3}\big)^2 = 3 - \frac{1}{3} = \frac{8}{3}$, that is $h = \frac{2\sqrt{2}}{\sqrt{3}}$. The centroid of the tetrahedron lies one quarter of the way along the altitude and so we have $r = \frac{1}{4}\, h = \frac{1}{\sqrt{6}}$. Thus the sides of S are of length $2\big(\frac{1}{r}+1\big) = 2(\sqrt{6}+1)$.

3: Let S be a set of 4 distinct real numbers. Show that there exist $a, b \in S$ such that

$$0 < \frac{a-b}{1+ab} \le 1\,.$$

Solution: Let $S = \{x_1, x_2, x_3, x_4\}$ with $x_1 < x_2 < x_3 < x_4$. Let $\theta_i = \tan^{-1}(x_i)$ for each i so we have $-\frac{\pi}{2} < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \frac{\pi}{2}$. Let $\phi_1 = \theta_2 - \theta_1$, $\phi_2 = \theta_3 - \theta_2$, $\phi_3 = \theta_4 - \theta_3$ and $\phi_4 = \pi + \theta_1 - \theta_4$. Then we have $0 < \phi_i$ for all i and we have $\phi_1 + \phi_2 + \phi_3 + \phi_4 = \pi$. It follows that $\phi_i \le \frac{\pi}{4}$ for some i. Choose k so that $\phi_k \le \frac{\pi}{4}$. Since $0 < \phi_k \le \frac{\pi}{4}$ we have $0 < \tan\phi_k \le 1$. Finally note that if $k \in \{1,2,3\}$ then we have

$$\tan\phi_k = \tan(\theta_{k+1} - \theta_k) = \frac{\tan\theta_{k+1} - \tan\theta_k}{1 + \tan\theta_{k+1}\tan\theta_k} = \frac{x_{k+1} - x_k}{1 + x_{k+1}x_k}$$

and if $k = 4$ then we have

$$\tan\phi_k = \tan(\pi + \theta_1 - \theta_4) = \tan(\theta_1 - \theta_4) = \frac{\tan\theta_1 - \tan\theta_4}{1 + \tan\theta_1\tan\theta_4} = \frac{x_1 - x_4}{1 + x_1x_4}\,.$$

4: Let $f(z)$ be a polynomial with complex coefficients of degree $n \geq 1$. Show that there exist at least $n+1$ distinct complex numbers z with $f(z) \in \{0,1\}$.

Solution: For a finite set A, let $|A|$ denote the number of elements in A. For a polynomial $f(x)$ and for $a \in \mathbf{C}$, let $\text{mult}(f,a)$ denote the smallest $m \in \text{N}$ such that $(x-a)^m$ is a factor of $f(x)$. (in particular $\text{mult}(f,a) = 0$ when a is not a root of f). For a polynomial $f(x) = \sum\limits_{k=0}^{n} c_k x^k$, let $f'(x)$ denote the derivative of f, which is given by $f'(x) = \sum\limits_{k=0}^{n} k\, c_k x^{k-1}$. As with real polynomials, we have the product rule $(fg)' = f'g + fg'$ and we have $\text{mult}(f,a) = \text{mult}(f',a) + 1$. By the Fundamental Theorem of Algebra, for complex polynomials (but not for real polynomials) we have $\sum\limits_{a \in \mathbf{C}} \text{mult}(f,a) = \deg(f)$. Let $g(x) = f(x) - 1$. Note that $g'(x) = f'(x)$. Let $A = \big\{a \in \mathbf{C} \big| f(a) = 0\big\}$ and let $B = \big\{b \in \mathbf{C} \big| f(b) = 1\big\} = \big\{b \in \mathbf{C} \big| g(b) = 0\big\}$. Note that $A \cap B = \emptyset$. Then

$$\begin{aligned}
|A| &= \sum_{a \in A} 1 = \sum_{a \in A} \big(\text{mult}(f,a) - \text{mult}(f',a)\big) \\
|B| &= \sum_{b \in B} 1 = \sum_{b \in B} \big(\text{mult}(g,b) - \text{mult}(g',b)\big) \\
|A| + |B| &= \sum_{a \in A} \text{mult}(f,a) + \sum_{b \in B} \text{mult}(g,b) - \sum_{a \in A} \text{mult}(f',a) - \sum_{b \in B} \text{mult}(f',b) \\
&\geq \sum_{a \in A} \text{mult}(f,a) + \sum_{b \in B} \text{mult}(g,b) - \sum_{c \in \mathbf{C}} \text{mult}(f',c) \\
&= \deg(f) + \deg(g) - \deg(f') = n + n - (n-1) = n+1.
\end{aligned}$$

5: Show that for every integer $n \geq 2$ there exists a finite group G with elements $a, b \in G$ such that $|a| = 2$, $|b| = 3$ and $|ab| = n$. (For $x \in G$, $|x|$ denotes the order of x in G).

Solution: Let S_m denote the m^{th} symmetric group (that is the group of permutations of $\{1, 2, \cdots, m\}$). Using cycle notation, in S_m the elements of order 2 are the products of disjoint 2-cycles and the element of order 3 are the products of disjoint 3-cycles. In S_m (where m is large enough so that the a_i can be distinct), we have

$$\begin{aligned}
(a_1, a_2) &= (a_2, a_3)(a_1, a_3, a_2) \\
(a_1, a_2, a_3) &= (a_1, a_4)(a_2, a_3)(a_1, a_3, a_4) \\
(a_1, a_2, a_3, a_4) &= (a_1, a_4)(a_1, a_2, a_3) \\
(a_1, a_2, a_3, a_4, a_5) &= (a_1, a_5)(a_2, a_3)(a_1, a_3, a_4) \\
(a_1, a_2, a_3, a_4, a_5, a_6) &= (a_1, a_6)(a_2, a_3)(a_4, a_5)(a_1, a_3, a_5).
\end{aligned}$$

We claim that for all $n, m \in \mathbf{Z}$ with $4 \leq n \leq m$, every n-cycle $\gamma = (a_1, a_2, \cdots, a_n) \in S_m$ can be expressed as a product $\gamma = \alpha\beta$ where α is a product of disjoint 2-cycles permuting only elements in $\{a_1, \cdots, a_n\}$ and where β is a product of disjoint 3-cycles permuting only elements in $\{a_1, \cdots, a_{n-1}\}$. We prove this by induction on n. As shown above, the claim is true when $n = 4, 5, 6$. Fix $n =\geq 7$ and suppose the claim holds for $n-3$. Let $\gamma = (a_1, a_2, \cdots, a_n) \in S_m$ where $m \geq n$. By the induction hypothesis, we can write $(a_2, a_3, \cdots, a_{n-2}) = \alpha\beta \in S_m$ where α is a product of disjoint 2-cycles permuting only elements in $\{a_2, \cdots, a_{n-2}\}$ and β is a product of disjoint 3-cycles permuting only elements in $\{a_2, \cdots, a_{n-3}\}$. Then we have

$$\gamma = (a_1, a_n)(a_2, a_3, \cdots, a_{n-2})(a_1, a_{n-2}, a_{n-1}) = \alpha'\beta'$$

where $\alpha' = (a_1, a_n)\alpha$ and $\beta' = \beta(a_1, a_{n-2}, a_{n-1})$.

6: Let $f : [0,1] \to [0,1]$ be continuous. Let $a_0 = 0$ and for $n \geq 1$ let $a_n = f(a_{n-1})$. Suppose $\lim\limits_{n\to\infty} (a_n - a_{n-1}) = 0$. Show that the sequence $\{a_n\}$ converges.

Solution: Suppose, for a contradiction, that $\{a_n\}$ diverges. Let $a = \liminf a_n$ and let $b = \limsup a_n$. Since $\{a_n\}$ diverges, we have $a < b$. Since the sequence $\{a_n\}$ oscillates back and forth, at times approaching a and at times approaching b, it follows, we claim, that there is some term $a_k \in \left[a + \frac{b-a}{3}, b - \frac{b-a}{3}\right]$. Here is a formal proof of this claim. Choose n_0 so that $n > n_0 \Longrightarrow |a_n - a_{n-1}| < \frac{b-a}{3}$ (we can do this since $|a_n - a_{n-1}| \to 0$). Choose $n_1 > n_0$ so that $a_{n_1} < a + \frac{b-a}{3}$ (we can do this since $\liminf a_n = a$). Choose $n_2 > n_1$ so that $a_{n_2} > b - \frac{b-a}{3}$ (we can do this since $\limsup a_n = b$). Choose k with $n_1 < k \leq n_2$ so that $a_{n_1}, a_{n_1+1}, \cdots, a_{k-1} < a + \frac{b-a}{3}$ and $a_k \geq a + \frac{b-a}{3}$. By our choice of k we have

$$a + \tfrac{b-a}{3} \leq a_k < a_{k-1} + \tfrac{b-a}{3} < a + \tfrac{b-a}{3} + \tfrac{b-a}{3} = b - \tfrac{b-a}{3},$$

proving the claim. Note that $f(a_k) \neq a_k$, since otherwise the sequence $\{a_n\}$ would become constant for $n \geq k$. Let $\epsilon = \frac{1}{3}|f(a_k) - a_k|$, and choose $\delta > 0$ with $\delta \leq \epsilon$ and $\delta \leq \frac{b-a}{3}$ $\big($so that $(a_k - \delta, a_k + \delta) \subseteq (a,b)\big)$ such that $|x - a_k| < \delta \Longrightarrow \big|f(x) - f(a_k)\big| < \epsilon$ (we can do this since f is continuous). Note that for $|x - a_k| < \delta$ we have $|x - a_k| < \epsilon$ and $\big|f(x) - f(a_k)\big| < \epsilon$ and so

$$3\epsilon = \big|f(a_k) - a_k\big| \leq \big|f(a_k) - f(x)\big| + \big|f(x) - x\big| + |x - a_k| < 2\epsilon + |f(x) - x|$$

and hence $\big|f(x) - x\big| > \epsilon$. Choose l_0 so that $n > l_0 \Longrightarrow |a_l - a_{l-1}| < \epsilon$. Since the sequence $\{a_n\}$ oscillates back and forth, at times approaching a and at times approaching b, it follows (as above) that we can choose $l > l_0$ with $a_l \in \big(a_k - \delta, a_k + \delta\big)$. This gives the desired contradiction because since $l > l_0$ we have $|f(a_l) - a_l| = |a_{l+1} - a_l| < \epsilon$ but since $|a_l - a_k| < \delta$ we have $|f(a_l) - a_l| > \epsilon$.