SPECIAL K
Saturday November 5, 2011
10:00 am - 1:00 pm

1: Find the number of sequences $a_1, a_2, \cdots, a_6$ with each $a_i \in \{1,2,3,4\}$ such that

$$a_1 < a_2 \ , \ a_2 > a_3, \ a_3 < a_4 \ , \ a_4 > a_5 \ , \ a_5 < a_6 \ \text{ and } \ a_6 > a_1 \, .$$

2: Find the number of solutions to the congruence $x^2 \equiv 40\,x \mod 10^6$.

3: Four spheres of radius 1 are inscribed in a regular tetrahedron so that each sphere is tangent to the other 3 spheres and to 3 of the faces of the tetrahedron. Find the length of the edges of the tetrahedron.

4: Let S be a set of 4 distinct real numbers. Show that there exist $a, b \in S$ such that

$$0 < \frac{a-b}{1+ab} \leq 1 \, .$$

5: Let $f(z)$ be a polynomial with complex coefficients of degree $n \geq 1$. Show that there exist at least $n+1$ distinct complex numbers z with $f(z) \in \{0, 1\}$.

6: Let $f : [0,1] \to [0,1]$ be continuous. Let $a_0 = 0$ and for $n \geq 1$ let $a_n = f(a_{n-1})$. Suppose that $\lim\limits_{n\to\infty} (a_n - a_{n-1}) = 0$. Show that the sequence $\{a_n\}$ converges.

BIG E
Saturday November 5, 2011
10:00 am - 1:00 pm

1: Find $\displaystyle\int_0^e x\, e^{-\lfloor \ln x \rfloor}\, dx$. (For $y \in \mathbf{R}$, $\lfloor y \rfloor$ denotes the largest integer n with $n \le y$).

2: Four spheres of radius 1 are inscribed in a regular tetrahedron so that each sphere is tangent to the other 3 spheres and to 3 of the faces of the tetrahedron. Find the length of the edges of the tetrahedron.

3: Let S be a set of 4 distinct real numbers. Show that there exist $a, b \in S$ such that

$$0 < \frac{a-b}{1+ab} \le 1\,.$$

4: Let $f(z)$ be a polynomial with complex coefficients of degree $n \ge 1$. Show that there exist at least $n+1$ distinct complex numbers z with $f(z) \in \{0,1\}$.

5: Show that for every integer $n \ge 2$ there exists a finite group G with elements $a, b \in G$ such that $|a| = 2$, $|b| = 3$ and $|ab| = n$. (For $x \in G$, $|x|$ denotes the order of x in G).

6: Let $f : [0,1] \to [0,1]$ be continuous. Let $a_0 = 0$ and for $n \ge 1$ let $a_n = f(a_{n-1})$. Suppose that $\lim_{n\to\infty} (a_n - a_{n-1}) = 0$. Show that the sequence $\{a_n\}$ converges.