SPECIAL K
Saturday November 6, 2010
10:00 am - 1:00 pm

1: Find the minimum possible discriminant $\Delta = b^2 - 4ac$ of a quadratic $f(x) = ax^2 + bx + c$ which satisfies the requirement that $f\big(f\big(f(0)\big)\big) = f(0)$.

2: Show that for every integer a, there exist infinitely many perfect powers of the form

$$a + 2010\,t\ ,\ \ t \in \mathbf{Z}\,.$$

(A *perfect power* is an integer of the form n^k for some integers $n \geq 0$ and $k \geq 2$).

3: Let n be a positive integer. Evaluate $\displaystyle\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

4: A point $p = (x,y)$ is chosen at random (with uniform distribution) in the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Find the probability that, in the triangle with vertices at $(0,0)$, $(1,0)$ and p, the angle at each vertex is at most $\frac{5\pi}{12}$.

5: Let x be an irrational number, and let M be a positive integer. Show that there exist integers a and b with $b > 0$ such that

$$\left| x - \tfrac{a}{b} \right| < \tfrac{1}{Mb}\,.$$

6: Let f be continuous on $[0,1]$ and differentiable in $(0,1)$. Suppose there exists $M > 0$ such that for all $x \in (0,1)$ we have $\big| f(0) - f(x) + x f'(x) \big| < Mx^2$. Prove that f is differentiable (from the right) at 0.

BIG E
Saturday November 6, 2010
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2: Show that for every integer a, there exist infinitely many perfect powers of the form

$$a + 2010\,t\ ,\ \ t \in \mathbf{Z}\,.$$

(A *perfect power* is an integer of the form n^k for some integers $n \geq 0$ and $k \geq 2$).

3: Evaluate $\displaystyle\sum_{n=0}^{\infty} \int_0^{\pi} (-1)^n \sin^{2n} x\; dx$.

4: Two points p and q are chosen at random (with uniform distribution) in the unit ball $x^2 + y^2 + z^2 \leq 1$. Find the probability that the triangle with vertices at p, q and the origin is an acute-angled triangle.

5: Let A be the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is $A_{i,j} = \frac{1}{i+j}$. Show that A is invertible.

6: Let f be continuous on $[0,1]$ and differentiable in $(0,1)$. Suppose there exists $M > 0$ such that for all $x \in (0,1)$ we have $\big|f(0) - f(x) + xf'(x)\big| < Mx^2$. Prove that f is differentiable (from the right) at 0.