Solutions to the Special K Problems, 2009

1: Determine the number of ways the digits $1, 2, 3, \cdots, 8$ can be arranged to form an 8-digit number which is divisible by 11.

Solution: In order for an arrangement to give a multiple of 11, the alternating sum of the digits must be a multiple of 11. Note that modulo 2, the alternating sum of the digits is equal to the sum of the digits which is equal to 0, so the alternating sum of the digits must be even. To get a multiple of 11, the alternating sum of the digits must be a multiple of 22. The maximum possible alternating sum is $(8+7+6+5)-(4+3+2+1) = 16$ and the minimum is -16, so the alternating sum must be equal to 0. Thus the sum of the 4 digits in the even positions and the sum of the 4 digits in the odd positions must both be equal to 18. Note that there are 4 ways to choose 4 of the given 8 digits so that the digit 1 is one of the 4 chosen digits and the sum of the 4 chosen digits is 18:

$$(1,2,7,8)\ ,\ (1,3,6,8)\ ,\ (1,4,5,8)\ ,\ (1,4,6,7)$$

For each of these 4 possibilities, the 4 chosen digits can occupy either the even or the odd positions, they can be arranged in $4! = 24$ ways, and the 4 remaining digits can also be arranged in $4! = 24$ ways. Thus the total number of ways to arrange the given digits to obtain a multiple of 11 is $2 \cdot 24 \cdot 24 = 1152$.

2: Find the largest integer n such that $x^8 - x^2$ is a multiple of n for every integer x.

Solution: Since $2^8 - 2^2 = 252$, we must have $n \big| 252$. Note that $252 = 2^2 \cdot 3^2 \cdot 7$. By Fermat's Little Theorem, for all x we have $x^6 \equiv x \pmod 7$, so $x^8 \equiv x^2 \pmod 7$. For $x \equiv 0$ or $2 \pmod 4$ we have $x^2 \equiv 0 \pmod 4$ so $x^k \equiv 0 \pmod 4$ for all $k \geq 2$, and for $x \equiv 1$ or $3 \pmod 4$ we have $x^2 \equiv x^4 \equiv x^6 \equiv x^8 \equiv 1 \pmod 4$. In either case, we have $x^8 \equiv x^2 \pmod 4$. Also, if $3 \not| x$ then by the Euler Fermat Theorem we have $x^6 \equiv 1 \pmod 9$ so $x^8 \equiv x^2 \pmod 9$, and if $3 \big| x$ then we have $x^2 \equiv 0 \pmod 9$ so $x^k \equiv 0 \pmod 9$ for all $k \geq 2$. In either case $x^8 \equiv x^2 \pmod 9$. Thus for every integer x we have $x^8 \equiv x^2$ modulo 7, 4 and 9, and so by the Chinese Remainder Theorem, $x^8 \equiv x^2 \pmod{252}$. Thus $n = 252$ is the largest such integer.

3: Let $a_1 = 1$ and for $n \geq 2$ let $a_n = 2a_{n-1} + n$. Find $\lim\limits_{n\to\infty} \dfrac{a_n}{2^n}$.

Solution: We have $a_1 = 1$, $a_2 = 2a_1 + 2 = 2 \cdot 1 + 2 = 4$ and $a_3 = 2 \cdot a_2 + 3 = 2 \cdot 4 + 3 = 11$. We guess that $a_n = A \cdot 2^n + Bn + C$ for some integers A, B and C. In particular we must have $2A + B + C = a_1 = 1$, $4A + 2B + C = a_2 = 4$ and $8A + 3B + C = a_3 = 11$. Solving these three equations gives $A = 2$, $B = -1$ and $C = -2$. We claim that $a_n = 2 \cdot 2^n - n - 2$. When $n = 1$, the claim is true. Let $k \geq 1$ and suppose the claim is true when $n = k$, that is suppose that $a_k = 2 \cdot 2^k - k - 2$. Then when $n = k+1$ we have

$$a_n = 2a_{n-1} + n = 2a_k + (k+1) = 2(2 \cdot 2^k - k - 2) + (k+1) = 2 \cdot 2^{k+1} - (k+1) - 2 = 2 \cdots 2^n - n - 2$$

so the claim is true when $n = k+1$, and hence by mathematical induction, the claim is true for all $n \geq 1$. Since, by l'Hôpital's Rule, $\lim\limits_{n\to\infty} \dfrac{n}{2^n} = \lim\limits_{x\to\infty} \dfrac{x}{2^x} = \lim\limits_{x\to\infty} \dfrac{1}{\ln 2 \cdot 2^x} = 0$, we have

$$\lim_{n\to\infty} \frac{a_n}{2^n} = \lim_{n\to\infty} \frac{2 \cdot 2^n - n - 2}{2^n} = \lim_{n\to\infty} \left(2 - \frac{n}{2^n} - \frac{1}{2^{n-1}} \right) = 2\,.$$

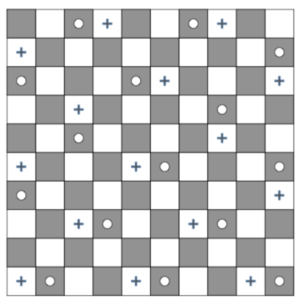
4: Let f and g be real-valued functions defined on $[0,1]$. Suppose that $f(0)>0$, $f(1)<0$, $f+g$ is increasing, and g is continuous on $[0,1]$. Show that $f(x)=0$ for some $x\in[0,1]$.

Solution: Let $S=\{x\in[0,1]\,|\,f(x)\geq 0\}$, and let $a=\inf S$. For all $x\in S$ we have $f(x)\geq 0$ and $a\geq x$, so since $f+g$ is increasing, we have $f(a)+g(a)\geq f(x)+g(x)\geq g(x)$. Since $g(x)\leq f(a)+g(a)$ for all $x\in S$, a is a limit point of S, and g is continuous, we have $g(a)\leq f(a)+g(a)$, and so $0\leq f(a)$. Thus $a\in S$.

Also, since $f(a)\geq 0$, $f+g$ is increasing and $f(1)<0$ we have $g(a)\leq f(a)+g(a)\leq f(1)+g(1)\leq g(1)$. Since g is continuous, by the Intermediate Value Theorem, we can choose $t\in[a,1]$ so that $g(t)=f(a)+g(a)$. Then we have $a\leq t\implies f(a)+g(a)\leq f(t)+g(t)\leq f(t)+f(a)+g(a)$ and so $0\leq f(t)$ so $t\in S$. Since $a\leq t\in S$ and $a=\inf S$, we have $a=t$, so $g(a)=g(t)=f(a)+g(a)$, and so $f(a)=0$.

5: Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal).

Solution: Colour the squares black and white in a checkerboard pattern. Each white square must be next to a coin on a black square. In the picture below, there are 15 white squares labeled with a cross. No coin can be next to two of these labeled squares, so there must be at least 15 coins on black squares. On the other hand, if coins are placed on the 15 black squares labeled with a small white circle, then every white square is next to a coin. Similarly, there must be at least 15 coins on white squares, and 15 such coins suffice. Thus the minimum number of coins is 30.



6: A set S of positive integers contains exactly 20 multiples of 2, exactly 20 multiples of 3, and exactly 20 multiples of 5. Show that there is a subset of S which contains exactly 10 multiples of 2, exactly 10 multiples of 3, and exactly 10 multiples of 5.

Solution: We claim that for for all $n\geq 2$, every set S of positive integers such that each of 2, 3 and 5 divides exactly n elements in S, has a subset $T\subset S$ such that each of 2, 3 and 5 divides exactly 2 elements of T. The claim is true when $n=2$ (we can take $T=S$). Let $m\geq 3$ and suppose the claim holds when $n=m-1$. Let S be a set of positive integers such that each of 2, 3 and 5 divides exactly m elements of S. Let $S_2=\{a\in S\,|\,2|a,3\nmid a,5\nmid a\}$, $S_3=\{a\in S\,|\,2\nmid a,3|a,5\nmid a\}$, $S_5=\{a\in S\,|\,2\nmid a,3\nmid a,5|a\}$, $S_{35}=\{a\in S\,|\,2\nmid a,3|a,5|a\}$, $S_{25}=\{a\in S\,|\,2|a,3\nmid a,5|a\}$, $S_{23}=\{a\in S\,|\,2|a,3|a,5\nmid a\}$ and $S_{235}=\{a\in S\,|\,2|a,3|a,5|a\}$. Let k_2, k_3, k_5, k_{35}, k_{25}, k_{23} and k_{235} be the number of elements in each of these sets. Since each of 2, 3 and 5 divides exactly m elements of S, we have $k_2+k_{25}+k_{23}+k_{235}=k_3+k_{35}+k_{23}+k_{235}=k_5+k_{35}+k_{25}+k_{235}=m$ and so $k_2-k_{35}=k_3-k_{25}=k_5-k_{23}$. Let $c=k_2-k_{35}=k_3-k_{25}=k_5-k_{23}$. We consider several cases. In the case that $c<0$ we have $k_{35}>0$, $k_{25}>0$ and $k_{23}>0$, so we can take $T\subset S$ to consist of one element from each of the sets S_{35}, S_{25} and S_{23}. In the case that $c=0$ and $k_2=k_3=k_5=0$, we also have $k_{35}=0$, $k_{25}=0$ and $k_{23}=0$, so we must have $k_{235}=m$, and so we can take $T\subset S$ to consist of two elements from the set S_{235}. In the case that $c=0$ and one of k_2, k_3 and k_5 is non-zero, say $k_2\neq 0$ so $k_{35}=k_2\neq 0$, we can choose one element from each of the sets S_2 and S_{35}, remove these two elements from S to obtain a set $R\subset S$ with the property that each of 2, 3 and 5 divides exactly $m-1$ elements of R, and then use the induction hypothesis to obtain $T\subset R$ such that each of 2, 3 and 5 divides exactly two elements of T. In the case that $c>0$ we have $k_2>0$, $k_3>0$ and $k_5>0$, so we can choose one element from each of the sets S_2, S_3 and S_5, remove these three elements from S to obtain a set $R\subset S$ with the property that each of 2, 3 and 5 divides exactly $m-1$ elements in R, and then use the induction hypothesis to obtain $T\subset R$ such that each of 2, 3 and 5 divides exactly 2 elements of T. Thus by Mathematical Induction, our claim is true for all $n\geq 2$. The given statement now follows by applying the proven claim 5 times.

Solutions to the Big E Problems, 2009

1: Find the largest integer n such that $x^8 - x^2$ is a multiple of n for every integer x.

Solution: Recall that (as a corollary of the Euler Fermat Theorem and the Chinese Remainder Theorem) for $n = {p_1}^{k_1}{p_2}^{k_2}\cdots{p_m}^{k_m}$ where the p_i are distinct primes and each k_i is a positive integer, if we let $\psi(n) = \text{lcm}\big(\phi(p_1), \phi(p_2), \cdots, \phi(p_m)\big)$ and $\kappa(n) = \max\big(k_1, k_2, \cdots, k_m\big)$ then for all integers k, l and x with $k, l \geq \kappa(n)$ we have $x^k \equiv x^l \pmod{n} \Longrightarrow k \equiv l \pmod{\psi(n)}$.

If $n\big|(x^8 - x^2)$ for every $x \in \mathbf{Z}$, then in particular $n\big|(2^8 - 2^2)$, that is $n\big|252$, so we have $n \leq 252$. On the other hand, if $n = 252 = 2^2 \cdot 3^2 \cdot 7$ then $\psi(n) = \text{lcm}\big(\phi(4), \phi(9), \phi(7)\big) = \text{lcm}(2, 6, 6) = 6$ and $\kappa(n) = \max(2, 2, 1) = 2$, and so we have $x^8 \equiv x^2 \pmod{n}$ by the above result.

2: Show that for all real numbers r and s, we have $r + s = 10$ if and only if there exist 2×2 matrices A and B, with real entries, such that A has eigenvalues 1 and 3, B has eigenvalues 2 and 4, and $A + B$ has eigenvalues r and s.

Solution: Recall that for a diagonalizable matrix, the sum of the eigenvalues of the matrix is equal to its trace, and the product of the eigenvalues of the matrix is equal to its determinant.

If A has eigenvalues 1 and 3, and B has eigenvalues 2 and 4, and $A + B$ has eigenvalues r and s, then we have $r + s = \text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) = (1 + 3) + (2 + 4) = 10$. Conversely, given r and s with $r + s = 10$, we choose $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and we wish to choose $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 2 + 4 = 6$, $ad - bc = 2 \cdot 4 = 8$, and $\det(A + B) = rs$. We have $A + B = \begin{pmatrix} 1 + a & b \\ c & 3 + d \end{pmatrix}$ so $\det(A + B) = 3 + d + 3a + ad - bc = 3 + d + 3a + 8 = 11 + 3a + d$, and so we need to solve $a + d = 6$ and $3a + d = rs - 11$. This pair of equations has solution $a = \frac{rs - 17}{2}$ and $d = \frac{29 - rs}{2}$. Finally, we choose b and c so that $bc = ad - 8$ to obtain the required matrix B.

3: A circle, on the surface of a sphere of surface area 1, divides the sphere into two parts. The smaller of these parts is removed and replaced by a hemisphere. The area of the resulting surface is $\frac{9}{8}$. Find the surface area of the hemisphere.

Solution: Recall that the area A of a spherical cap of height h on a sphere of radius r is given by $A = 2\pi rh$ (more generally, the area A of any slice of thickness h on a sphere of radius r is given by $A = 2\pi rh$).

Let r be the radius of the given sphere, let y be the radius of the given circle on the sphere, and let $x = \sqrt{r^2 - y^2}$. The area of the sphere is $4\pi r^2 = 1$. The part that is removed is a spherical cap of height $h = (r - x)$ so it has area $2\pi r(r - x)$. The added hemisphere has radius y, so it has area $2\pi y^2$. The area of the resulting surface is $1 - 2\pi r(r - x) + 2\pi y^2 = \frac{9}{8}$ so we have $2\pi(y^2 - r^2) + 2\pi rx = \frac{1}{8}$. Since $y^2 - r^2 = -x^2$ and $4\pi r^2 = 1$ so $2\pi r = \sqrt{\pi}$, this gives

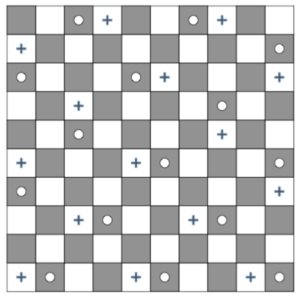
$$-2\pi x^2 + \sqrt{\pi}\, x - \tfrac{1}{8} = 0 \Longrightarrow 4\pi x^2 - 2\sqrt{\pi}\, x + \tfrac{1}{4} = 0 \Longrightarrow \big(2\sqrt{\pi} - \tfrac{1}{2}\big)^2 = 0$$

$$\Longrightarrow x = \frac{1}{4\sqrt{\pi}} \Longrightarrow x^2 = \frac{1}{16\pi} \Longrightarrow y^2 = r^2 - x^2 = \tfrac{1}{4\pi} - \tfrac{1}{16\pi} = \tfrac{3}{16\pi}\,.$$

Thus the area of the hemisphere is $2\pi y^2 = \frac{3}{8}$.

4: Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal).

Solution: Colour the squares black and white in a checkerboard pattern. Each white square must be next to a coin on a black square. In the picture below, there are 15 white squares labeled with a cross. No coin can be next to two of these labeled squares, so there must be at least 15 coins on black squares. On the other hand, if coins are placed on the 15 black squares labeled with a small white circle, then every white square is next to a coin. Similarly, there must be at least 15 coins on white squares, and 15 such coins suffice. Thus the minimum number of coins is 30.



5: Let n be a positive integer and let $p_1, p_2, \cdots, p_n$ be non-constant polynomials with integer coefficients. Show that there exists a positive integer k such that each $p_i(k)$ is composite.

Solution: Recall that if p is a polynomial with integer coefficients then for all integers a, k and l we have $k \big| l \implies k \big| \left(p(a+l) - p(a)\right)$, and in particular, for all integers a and k we have $p(a) \big| p\big(a + kp(a)\big)$.

Note that each p_i takes the values $0, \pm 1$ finitely many times. Choose an integer a so that for all i we have $p_i(a) \notin \{0, \pm 1\}$. Let $l = \operatorname{lcm}\big(p_1(a), p_2(a), \cdots, p_n(a)\big)$. By the above result, for each p_i and for all integers k we have $p_i(a) \big| p_i(a + kl)$. Note that each p_i takes the values $0, \pm p_i(a)$ finitely many times. Choose k so that each $p_i(a + kl) \notin \{0, \pm p_i(a)\}$. Then each $p_i(a + kl)$ is composite, with $p_i(a)$ as a factor.

6: Let $a_0 = 1$ and for $n \geq 0$ let $a_{n+1} = a_n - \frac{1}{2}{a_n}^2$. Find $\lim\limits_{n \to \infty} n\, a_n$, if it exists.

Solution: Recall that for any sequence $\{b_n\}$, if $\lim\limits_{n \to \infty} b_n = l$ then $\lim\limits_{n \to \infty} \dfrac{b_1 + b_2 + \cdots + b_n}{n} = l$.

For all $n \geq 0$ we have $a_{n+1} = a_n - \frac{1}{2}{a_n}^2 \leq a_n$, so $\{a_n\}$ is non-increasing. Also note that $0 < a_0 < 2$ and that if $0 < a_n < 2$ then $0 < a_n - \frac{1}{2}{a_n}^2 \leq \frac{1}{2} < 2$, so by induction we have $0 < a_n < 2$ for all $n \geq 0$, and so $\{a_n\}$ is bounded below, by 0. Thus $\{a_n\}$ converges by the Monotone Convergence Theorem. Let $l = \lim\limits_{n \to \infty} a_n$. By taking the limit on both sides of the formula $a_{n+1} = a_n - \frac{1}{2}{a_n}^2$, we se that $l = l - \frac{1}{2}l^2$, and hence $l = 0$. Thus $\lim\limits_{n \to \infty} a_n = 0$.

For $n \geq 1$, let $b_n = \frac{1}{a_n} - \frac{1}{a_{n-1}}$. Note that since $a_n \to 0$ as $n \to \infty$ we have

$$b_{n+1} = \frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{1}{a_n - \frac{1}{2}{a_n}^2} - \frac{1}{a_n} = \frac{1 - \left(1 - \frac{1}{2}a_n\right)}{a_n - \frac{1}{2}{a_n}^2} = \frac{\frac{1}{2}a_n}{a_n - \frac{1}{2}{a_n}^2} = \frac{\frac{1}{2}}{1 - \frac{1}{2}a_n} \to \frac{1}{2} \text{ as } n \to \infty\,.$$

Thus $\lim\limits_{n \to \infty} b_n = \frac{1}{2}$. Also note that $\dfrac{b_1 + b_2 + \cdots + b_n}{n} = \dfrac{\frac{1}{a_n} - \frac{1}{a_0}}{n} = \frac{1}{na_n} - \frac{1}{na_0}$, so by the above result we have $\lim\limits_{n \to \infty} \left(\frac{1}{na_n} - \frac{1}{na_0}\right) = \frac{1}{2}$. Since $\lim\limits_{n \to \infty} \frac{1}{na_0} = 0$, we have $\lim\limits_{n \to \infty} \frac{1}{na_n} = \frac{1}{2}$. Thus $\lim\limits_{n \to \infty} na_n = 2$.