SPECIAL K
Saturday November 7, 2009
9:00 am - 12:00 noon

1: Determine the number of ways the digits $1, 2, 3, \cdots, 8$ can be arranged to form an 8-digit number which is divisible by 11.

2: Find the largest integer n such that $x^8 - x^2$ is a multiple of n for every integer x.

3: Let $a_1 = 1$ and for $n \geq 2$ let $a_n = 2a_{n-1} + n$. Find $\lim_{n \to \infty} \frac{a_n}{2^n}$.

4: Let f and g be real-valued functions defined on $[0, 1]$. Suppose that $f(0) > 0$, $f(1) < 0$, $f + g$ is increasing, and g is continuous on $[0, 1]$. Show that $f(x) = 0$ for some $x \in [0, 1]$.

5: Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal).

6: A set S of positive integers contains exactly 20 multiples of 2, exactly 20 multiples of 3, and exactly 20 multiples of 5. Show that there is a subset of S which contains exactly 10 multiples of 2, exactly 10 multiples of 3, and exactly 10 multiples of 5.

BIG E
Saturday November 7, 2009
9:00 am - 12:00 noon

1: Find the largest integer n such that $x^8 - x^2$ is a multiple of n for every integer x.

2: Show that for all real numbers r and s, we have $r + s = 10$ if and only if there exist 2×2 matrices A and B, with real entries, such that A has eigenvalues 1 and 3, B has eigenvalues 2 and 4, and $A + B$ has eigenvalues r and s.

3: A circle, on the surface of a sphere of surface area 1, divides the sphere into two parts. The smaller of these parts is removed and replaced by a hemisphere. The area of the resulting surface is $\frac{9}{8}$. Find the surface area of the hemisphere.

4: Coins are placed on some of the 100 squares in a 10×10 grid. Every square is next to another square with a coin. Find the minimum possible number of coins. (We say that two squares are next to each other when they share a common edge but are not equal).

5: Let n be a positive integer and let $p_1, p_2, \cdots, p_n$ be non-constant polynomials with integer coefficients. Show that there exists a positive integer k such that each $p_i(k)$ is composite.

6: Let $a_0 = 1$ and for $n \geq 0$ let $a_{n+1} = a_n - \frac{1}{2}{a_n}^2$. Find $\lim\limits_{n\to\infty} n\, a_n$, if it exists.