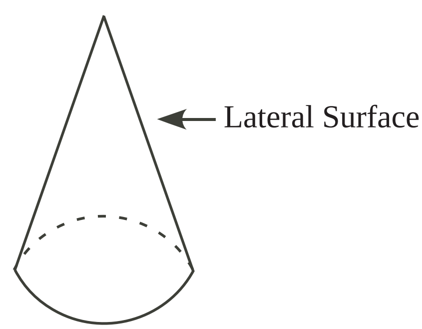
SPECIAL K
Saturday 04 November 2006
9 a.m. to 12 noon

1. A party of 100 mathies went to the circus. The total charge for admission was \$95. For faculty, the charge was \$10. For grad students, the charge was \$2.50. For undergrads, the charge was \$0.50. Determine the number of faculty, grad students and undergrads who went to the circus, and explain how you got your answer.

2. A complex number $z = a + bi$ with $a, b \in \mathbb{Z}$ is called a Gaussian integer. A Gaussian integer z is called *even* if $1 + i$ divides exactly into z, giving a quotient which is a Gaussian integer. Prove that a Gaussian integer $z = a + bi$ is even if and only if a and b have the same parity.

3. Let $M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$. If $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a Pythagorean triple with $x^2 + y^2 = z^2$, prove that $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = M\mathbf{x}$ is also a Pythagorean triple and that $\gcd(a, b, c) = \gcd(x, y, z)$.

4. The *lateral* surface area of a cone is the surface area of the cone that does not include the circular base. Among all circular cones with lateral surface area equal to K, determine the cone with largest possible volume and its volume.



5. Let d be a positive integer and let s be a string of d decimal digits ending in 1, 3, 7 or 9. Prove that there is a positive integer n such that n^3 ends in s.

6. Prove that there does not exist a rational function $f(x)$ with real coefficients such that

$$f\left(\frac{x^2}{x+1}\right) = p(x)$$

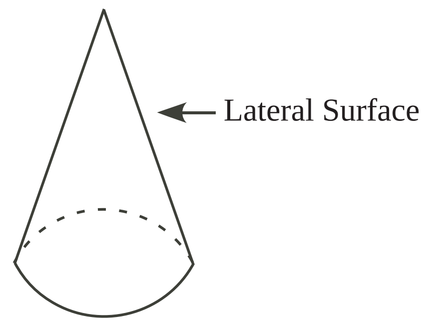
where $p(x)$ is a non-constant polynomial with real coefficients.

BIG E
Saturday 04 November 2006
9 a.m. to 12 noon

1. A party of 100 mathies went to the circus. The total charge for admission was \$95. For faculty, the charge was \$10. For grad students, the charge was \$2.50. For undergrads, the charge was \$0.50. Determine the number of faculty, grad students and undergrads who went to the circus, and explain how you got your answer.

2. Let k and n be positive integers with $k < n$. Determine the number of permutations of $\{1, 2, 3, \ldots, n\}$ in which $1, 2, \ldots, k$ appears as a subsequence, but $1, 2, \ldots, k, k + 1$ does not.

3. The *lateral* surface area of a cone is the surface area of the cone that does not include the circular base. Among all circular cones with lateral surface area equal to K, determine the cone with largest possible volume and its volume.



4. Evaluate $\displaystyle\int_0^1 \log x \log(1 - x)\, dx$.

5. Prove that there does not exist a rational function $f(x)$ with real coefficients such that

$$f\left(\frac{x^2}{x+1}\right) = p(x)$$

   where $p(x)$ is a non-constant polynomial with real coefficients.

6. Suppose that $A = \{a_1, \ldots, a_{2006}\}$ is a set of distinct integers. Consider an arbitrary sequence $b_1, \ldots, b_N$ where each $b_j \in A$. Prove that if $N \geq 2^{2006}$, then some (non-empty) product of consecutive terms in this sequence is a perfect square. Show by example that if $N < 2^{2006}$, then the result does not necessarily hold.