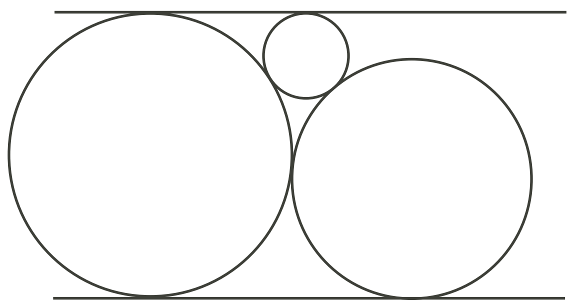
SPECIAL K
Saturday 29 October 2005
9 a.m. to 12 noon

1. Suppose that A and B are points on the parabola $y = x^2$, with ℓ the tangent line to the parabola which is parallel to AB. Show that the midpoint of AB, the point at which ℓ is tangent to the parabola, and the point of intersection of the tangent lines to the parabola at A and B are collinear.

2. While Lino was simplifying the fraction $\dfrac{A^3+B^3}{A^3+C^3}$ he cancelled the threes $\dfrac{A^{\not 3}+B^{\not 3}}{A^{\not 3}+C^{\not 3}}$ to obtain the fraction $\dfrac{A+B}{A+C}$. If $B \neq C$, determine a necessary and sufficient condition on A, B and C for Lino's method to actually yield the correct answer, ie. for $\dfrac{A^3+B^3}{A^3+C^3} = \dfrac{A+B}{A+C}$.

3. In the diagram, each of the three circles is tangent to the other two circles. Also, the large circle is tangent to the two lines, and each of the smaller circles is tangent to one of the lines. If the lines are parallel, and the radii of the two smaller circles are 4 and 9, determine the radius of the large circle.



4. Let n be a positive integer. Determine the value of $\displaystyle\prod_{k=0}^{n} \frac{(2k)!(2k)!}{k!(n+k)!}$.

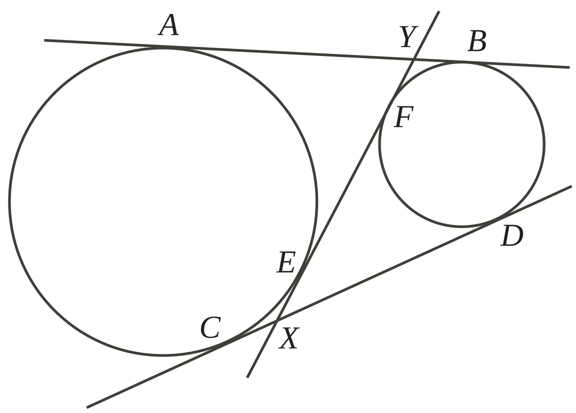
5. If a and b are positive real numbers, determine the value of $\displaystyle\lim_{n\to\infty} \left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n$.

6. If the decimal expansion of $x \in (0,1)$ can be written as $x = 0.x_1x_2x_3\cdots$, not ending in an infinite string of 9s, then define $f(x) = 0.x_2x_4x_6\cdots$ Determine the values of $x \in (0,1)$ at which $f(x)$ is not continuous.

BIG E
Saturday 29 October 2005
9 a.m. to 12 noon

1. While Lino was simplifying the fraction $\dfrac{A^3+B^3}{A^3+C^3}$ he cancelled the threes $\dfrac{A^{\not 3}+B^{\not 3}}{A^{\not 3}+C^{\not 3}}$ to obtain the fraction $\dfrac{A+B}{A+C}$. If $B \neq C$, determine a necessary and sufficient condition on A, B and C for Lino's method to actually yield the correct answer, ie. for $\dfrac{A^3+B^3}{A^3+C^3} = \dfrac{A+B}{A+C}$.

2. In the diagram, the lines through A and B, through C and D, and through E and F are tangent to both circles. Prove that $XE = YF$.



3. Let A be the 10×10 matrix induced by the 10 by 10 multiplication table, that is

$$A = \begin{pmatrix} 1 & 2 & \cdots & 10 \\ 2 & 4 & \cdots & 20 \\ \vdots & & & \vdots \\ 10 & 20 & \cdots & 100 \end{pmatrix}$$

Determine the eigenvalues of A and their multiplicities.

4. Suppose $x_1 = 0$, $x_2 = 1$ and $x_n = (n-1)(x_{n-1} + x_{n-2})$ for $n \geq 3$. Determine $\lim\limits_{n\to\infty} \dfrac{x_n}{n!}$.

5. Suppose $x \in (0,1)$, and write the decimal expansion of x as $x = 0.x_1x_2x_3\cdots$, not ending in an infinite string of 9s. Define the function $f : (0,1) \to \mathbb{R}$ by $f(x) = 0.x_2x_4x_6\cdots$ Determine all values of $x \in (0,1)$ at which $f(x)$ is not continuous.

6. Let $p(x)$ be a polynomial with rational coefficients. Suppose that $p(x)$ is always irrational when x is irrational. Prove that $p(x)$ is linear.