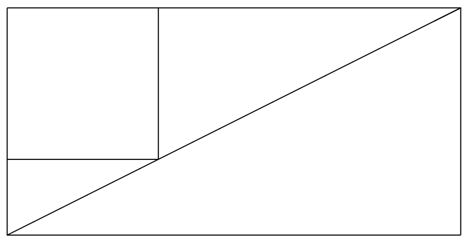
SPECIAL K
Saturday 06 November 2004
9 a.m. to 12 noon

1. Determine all possible pairs (x, y) which satisfy

$$\begin{aligned} x + y + \sqrt{x+y} &= 56 \\ x - y + \sqrt{x-y} &= 30 \end{aligned}$$

2. A square is drawn inside a rectangle of length a and width b, with one vertex of the square on the diagonal of the rectangle, as shown. If the square has side h, prove that $\dfrac{1}{h} = \dfrac{1}{a} + \dfrac{1}{b}$.



3. A function $f(x)$ is called periodic if there exists a p so that $f(x) = f(x+p)$ for all x.
Let $A \subseteq \mathbb{Z}$ and define $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$
We call the set A periodic if $\chi_A(x)$ is periodic.
Prove that if $S, T \subseteq \mathbb{Z}$ are periodic, then $S \cup T$ is periodic.

4. Let $a, b \in \mathbb{R}$ with $a, b > 0$. Define $a_0 = a$, $b_0 = b$ and

$$\begin{aligned} a_{n+1} &= \frac{2a_n b_n}{a_n + b_n} \\ b_{n+1} &= \frac{a_n + b_n}{2} \end{aligned}$$

for $n \geq 0$. Prove that $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ both exist and equal $\sqrt{ab}$.

5. Rectangle $ABCD$ has area 1 and is partitioned into mn congruent rectangles by $m-1$ horizontal lines and $n-1$ vertical lines. If the average area of *all* of the rectangles formed is $\frac{1}{7}$, determine all possible values of m and n.

6. Determine, with proof, the value of $\displaystyle\sum_{k=-\infty}^{\infty} f(k, 2004)$, where $f : \mathbb{Z} \times \mathbb{Z}_0^+ \to \{0, 1\}$ is defined by

$$f(k,0) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \qquad f(k, l+1) = \begin{cases} 1 & \text{if } f(k-1,l) + f(k,l) + f(k+1,l) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(Note: $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$.)

BIG E
Saturday 06 November 2004
9 a.m. to 12 noon

1. $ABCD$ is a rectangle, P is the midpoint of AB, and Q is the point on PD such that CQ is perpendicular to PD. Prove that triangle BQC is isosceles.

2. Let N be the integer whose base 10 representation is $11\cdots1122\cdots225$, where there are 2004 1's and 2005 2's in total. Prove that N is a perfect square.

3. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = K > 0$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, prove that $\lim_{x\to\infty} \frac{\ln(f(x))}{\ln(g(x))} = 1$.

4. The positive divisors of a positive integer n are written in increasing order starting with 1:

$$1 = d_1 < d_2 < d_3 < \cdots < n$$

   Find the number n if it is known that exactly three of the positive divisors are prime numbers, $n = d_{13} + d_{14} + d_{15}$, and $(d_5 + 1)^3 = d_{15} + 1$.

5. Let $n \in \mathbb{Z}^+$. Show that it is possible to find n points which lie on a circle such that the distance between any two of the points is an integer.

6. Determine, with proof, the value of $\displaystyle\sum_{k=-\infty}^{\infty} f(k, 2004)$, where $f : \mathbb{Z} \times \mathbb{Z}_0^+ \to \{0, 1\}$ is defined by

$$f(k,0) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \qquad f(k,l+1) = \begin{cases} 1 & \text{if } f(k-1,l) + f(k,l) + f(k+1,l) = 1 \\ 0 & \text{otherwise} \end{cases}$$

   (Note: $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$.)