SPECIAL K
Saturday November 1, 2003
9:00 am - 12:00 noon

1: How many times between midday and midnight is the hour hand of a clock at right angles to the minute hand?

2: At the vertices of a cube are written eight distinct positive integers. On each of the edges of the cube is written the greatest common divisor of the numbers at the endpoints of the edge. Prove or disprove: the numbers can be chosen so that the sum of the numbers at the vertices is equal to the sum of the numbers on the edges.

3: (a) Suppose we have six sticks with distinct lengths with the property that no matter which order the endpoints are joined (so that exactly three ends meet) they can always form the frame of a tetrahedron. How many tetrahedra can be formed by these six sticks? We shall assume that tetrahedra which are reflections of each other count as the same.

(b) Suppose we are given three positive numbers a, b, c such that the sum of any two is greater than the third. When is it possible to construct a tetrahedron with the property that the three edges which border any triangular face have lengths a, b and c?

4: Let $A_1, \cdots, A_n$ be a collection of n arcs on a sphere of radius ρ, where each arc A_i is an arc of a great circle C_i. Let $|A_i|$ denote the length of A_i. Suppose that $\sum_{i=1}^{n} |A_i| < \pi\rho$. Prove that there is some great circle C of the sphere which does not intersect any of the arcs.

5: For which positive integers n does there exist a smooth curve C and distinct points $P_1, \cdots, P_n$ in the plane such that the following three conditions are all satisfied:

(a) the curve passes through each point exactly once;

(b) the curve only ever crosses itself at right angles, and

(c) if C_i is the subset of the curve beginning at P_i and ending at P_{i+1}, then any two segments C_j and C_k intersect in exactly one point. Here, we adopt the convention that $P_{n+1} = P_1$, and that C_i includes its endpoints P_i and P_{i+1}.

BIG E
Saturday November 1, 2003
9:00 am - 12:00 noon

1: The hands of an accurate clock have lengths 3 and 4. Find the distance between the ends of the hands when that distance is increasing most rapidly.

2: Let A and B be $n \times n$ real matrices such that $A^n = B^n = 0$ and $AB = BA$. Prove that $(A+B)^n = 0$.

3: Let $x_1, x_2, \cdots$ be a sequence of real numbers such that $\sum_{i=1}^{\infty} x_i = 2003$. Determine all possible limiting values as $n \to \infty$ of $\frac{1}{n} \sum_{i=1}^{n} i\, x_i$.

4: Let $A_1, \cdots, A_n$ be a collection of n arcs on a sphere of radius ρ, where each arc A_i is an arc of a great circle C_i. Let $|A_i|$ denote the length of A_i. Suppose that $\sum_{i=1}^{n} |A_i| < \pi\rho$. Prove that there is some great circle C of the sphere which does not intersect any of the arcs.

5: For which positive integers n does there exist a smooth curve C and distinct points $P_1, \cdots, P_n$ in the plane such that the following three conditions are all satisfied:

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