SPECIAL K
Saturday November 2, 2002
9:00 am - 12:00 noon

1: Xavier and Yolanda play a game on a board which consists of a narrow strip which is one square wide and n squares long. They take turns at placing counters, which are one square wide and two squares long, on unoccupied squares on the board. The first player who cannot go loses. Xavier always plays first and both players always make the best available move.

(a) Who wins the game on a 4×1 board? Explain how they must play to win and why they are certain to win.

(b) Who wins the game on a 6×1 board? How?

(c) Who wins the game on an 8×1 board? How?

2: In a hockey league with 5 teams, each team plays each other team exactly once. Each game can end in a win (W), a loss (L), or a tie (T). Games played (GP), goals for (GF) and goals against (GA) are also tabulated. At some point during the season, the standings are as follows:

Team	GP	W	L	T	GF	GA
A			1	0	1	2
B			1			5
C	2		1		7	4
D		1		0	2	2
E		2		0	5	2

Determine the complete table, justifying your work.

3: Determine for each positive integer n the first digit after the decimal point in $\sqrt{n^2+n+1}$.

4: Show that for every positive integer n,

$$\frac{1}{2(n+1)} < \int_0^{\pi/4} \tan^n x \, dx < \frac{1}{2n} .$$

5: In a convex quadrilateral $ABCD$, the diagonals meet at the point S. Suppose that $\angle BAC = \angle DBC = \frac{\pi}{6}$ and $\angle ADB = \angle ACD = \frac{\pi}{4}$. Determine all possibilities for $\angle DSA$.

6: We write $f(n) \sim g(n)$ when $\lim\limits_{n\to\infty} \dfrac{f(n)}{g(n)} = 1$. For a positive integer n, define $d(n)$ to be the number of positive divisors of n. Prove that

$$\frac{d(1)+d(2)+\cdots+d(n)}{n} \sim \ln n .$$

BIG E
Saturday November 2, 2002
9:00 am - 12:00 noon

1: The positive integers $1, 4, 7, 10, \cdots$ are all of the form $3k+1$. They are grouped consecutively as follws

$$\{1\}\ ,\quad \{4,7\}\ ,\quad \{10,13,16\}\ ,$$

and so on, so that the n^{th} group has n numbers. Find the formula for the sum of the numbers in the n^{th} group.

2: The product

$$(1-a)(1-b)(1-c)(1-d)\cdots$$

is expanded in right-to-left lexicographic order (i.e, dictionary order, but reading from right to left) as

$$1 - a - b + ab - c + ac + bc - abc - d \cdots .$$

What is the sign on the 2002^{nd} term?

3: Let $f_n(x)$, $n \geq 0$ be a sequence of polynomials with $f_0 \equiv 1$ and

$$(n+1) f_{n+1}(x) = {f_n}'(x) + x\, f_n(x)\,.$$

Evaluate

$$f_0(1) - f_1(1) + f_2(1) - f_3(1) + \cdots .$$

4: Determine all non-negative integer solutions to the equation

$$(xy-7)^2 = x^2 + y^2\,.$$

5: Suppose A and B are real $n \times n$ symmetric matrices such that $AB = BA$. For each pair of $n \times 1$ column vectors v and w, we let $d(v, w)$ be the dimension of the space spanned by the six vectors v, w, Av, Aw, Bv and Bw. Find the minimum value of $d(v, w)$ as v and w range over all possible non-zero vectors.

6: Suppose we are given n points in the plane with the property that the area of the triangle formed by any three of them is at most 1. Prove that the points lie on the boundary or inside some triangle of area at most 4.