SPECIAL K
Saturday October 28, 2000
9:00 am - 12:00 noon

1: Let $x > 1$ be a real number, and $n > 1$ be an integer. Prove that

$$\sqrt[n]{x} \; < \; 1 + \frac{x-1}{n} \,.$$

2: Find the smallest (by area) right-angled triangle with integral sides in which a square with integral sides can be inscribed so that an angle of the square coincides with the right angle of the triangle.

3: Let S be a set of points in the plane. A circle C is said to be *framed* by S if C has a diameter whose endpoints both lie in S. Find all sets S of four points in the plane such that, for any two circles C_1 and C_2 framed by S, the set $S \cap C_1 \cap C_2$ is non-empty.

4: Let f be a real-valued continuous function of a real variable with the property that

$$\lim_{x \to +\infty} f\big(f(x)\big) = +\infty \;\text{ and }\; \lim_{x \to -\infty} f\big(f(x)\big) = -\infty \,.$$

Prove that $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ both exist and are infinite.

5: Peter tells Ian and Christopher that x and y are two integers with $1 < x < y$ and $x+y \leq 30$. Peter then gives Christopher the value of $x + y$ and Ian the value of xy.

(1) Ian says "I don't know the values of x and y."

(2) Christopher replies "I knew that you didn't know their values."

(3) Ian responds "I still don't know the values of x and y."

(4) Christopher exclaims "In that case, I know their values!"

What is the value of xy?

BIG E
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9:00 am - 12:00 noon

1: Define $Q_k = \frac{1}{(k+2)!} + \frac{2}{(k+3)!} + \frac{3}{(k+4)!} + \cdots$. Show that Q_0 is rational, but that Q_k is irrational for every positive integer k.

2: Let S be a set of points in the plane. A circle C is said to be *framed* by S if C has a diameter whose endpoints both lie in S. Find all sets S of four points in the plane such that, for any two circles C_1 and C_2 framed by S, the set $S \cap C_1 \cap C_2$ is non-empty.

3: Let f be a real-valued continuous function of a real variable with the property that

$$\lim_{x \to +\infty} f\big(f(x)\big) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} f\big(f(x)\big) = -\infty\,.$$

Prove that $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ both exist and are infinite.

4: Let a and b be non-zero complex numbers which satisfy the equation

$$a\,2^{|a|} + b\,2^{|b|} = (a+b)\,2^{|a+b|}\,.$$

Prove that $a^6 = b^6$.

5: Find the value of the infinite product $\prod_{n=1}^{\infty} \left(1 + \frac{1}{a_n}\right)$ where $a_1 = 1$ and $a_n = n(a_{n-1} + 1)$ for all $n \geq 2$.