SPECIAL K
Saturday October 31, 1998
9:00 am - 12:00 noon

1: Find the ratio of the area of a given triangle ABC to a triangle DEF whose sides have the same lengths as the medians of the triangle ABC.

2: A 5×5 square is divided into 25 unit squares. One of the numbers 1, 2, 3, 4, 5 is inserted into each of the unit squares in such a way that each row, each column and each of the two main diagonals contains each of the five numbers once and only once. The sum of the numbers in the four squares immediately below the diagonal from top left to bottom right is called the *score*. What is the highest possible score?

3: Find all pairs of integers (x, y) such that

$$(x+1)(x+2)(x+3) + x(x+2)(x+3) + x(x+1)(x+3) + x(x+1)(x+2) = y^{2^x}\ .$$

4: Does there exist an infinite sequence of positive real numbers $x_1, x_2, x_3, \cdots$ such that

$$x_{n+2} = \sqrt{x_{n+1}} - \sqrt{x_n}$$

for all $n \geq 1$?

5: Let $\mathbf{Q}$ denote the set of rational numbers. Find all functions $f : \mathbf{Q} \to \mathbf{Q}$ which satisfy the functional equation

$$f\big(x + f(y)\big) = f(x)f(y)$$

for all rational x and y.

BIG E
Saturday October 31, 1998
9:00 am - 12:00 noon

1: Solve the equation

$$x = \prod_{n=0}^{\infty} \left(1 + x^{-2^n}\right)$$

for $x > 1$.

2: A 5×5 square is divided into 25 unit squares. One of the numbers 1, 2, 3, 4, 5 is inserted into each of the unit squares in such a way that each row, each column and each of the two main diagonals contains each of the five numbers once and only once. The sum of the numbers in the four squares immediately below the diagonal from top left to bottom right is called the *score*. What is the highest possible score?

3: Evaluate

$$\sum_{(p,q)=1} \frac{1}{x^{p+q} - 1}$$

where the sum extends over all positive integers p and q such that p and q are relatively prime.

4: Does there exist an infinite sequence of positive real numbers $x_1, x_2, x_3, \cdots$ such that

$$x_{n+2} = \sqrt{x_{n+1}} - \sqrt{x_n}$$

for all $n \geq 1$?

5: Prove that there does not exist a function $f : (0, \infty) \to (0, \infty)$ satisfying the equation

$$f\left(f(x) + \frac{1}{f(x)}\right) = x + a$$

for all x, where $0 \leq a \leq 1$.